Mathematical Structuralism, S02

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December 10, 2020

1 Category Theory

Definition 1.1. A category $\mathcal{C}$ is the following data:

- a colection of objects, denoted by $ob(\mathcal{C})$,
- a collection of morphisms, denoted by $Mor(\mathcal{C})$,
- for any morphism $f \in Mor(\mathcal{C})$, an object $s(f)$ called the source of f,
- for any morphism $f \in Mor(\mathcal{C})$, an object $t(f)$ called the target of f,
- for any object $A \in ob(\mathcal{C})$, a morphism id_A,
- for any two morphisms $f, g \in Mor(\mathcal{C})$ such that $s(f) = t(g)$, a morphism $f \circ g$,

satisfying the following properties:

- $s(id_A) = t(id_A) = A$,
- $s(f \circ g) = s(g)$ and $t(f \circ g) = t(f)$,
- $f \circ id_A = f = id_B \circ f$, if $s(f) = A$ and $t(f) = B$,
- $f \circ (g \circ h) = (f \circ g) \circ h$.

For any $f \in Mor(\mathcal{C})$, we summarize the data $s(f) = A$ and $t(f) = B$ by $f : A \to B$. For any two objects $A, B \in ob(\mathcal{C})$ by $\mathcal{C}(A, B)$ or $Hom_{\mathcal{C}}(A, B)$, we mean the collection of all morphisms $f : A \to B$. A category is called small if $Mor(\mathcal{C})$ is a set. It is called locally small if $Hom_{\mathcal{C}}(A, B)$ is a set, for any two objects A, B.

$$
\begin{tikzcd}
& & D \\
A \arrow[rru, bend left, "jf=jg=jh"] \arrow[r, bend left, "f"] \arrow[r, "g" description] \arrow[r, bend right, "h"'] \arrow[dr, bend right, "if=ig=ih"'] & B \arrow[ur, "j"] \arrow[d, "i"] & \\
& C \arrow[uur, dotted, "k"'] &
\end{tikzcd}
$$

Philosophical Note 1.2. To have some informal interpretation in mind, read objects as the entities of a given discourse and maps as the transformations between them, composition as the composition of the transformations and the identity as the do-nothing transformation. Note that in a category, an object is just an abstract node that bears no information except what is encoded in the maps starting from or ending in the object itself. In this sense, the only way to inspect an object is by using its behaviour in the context of the other objects, other than that, it is just one abstract node.

Example 1.3. The collection of all sets as the objects and the usual functions as the morphisms with the usual composition and identity constitutes a category. This category is denoted by **Set**. If we restrict ourselves to the finite sets, then the result is the category **FinSet**.

Example 1.4. The collection of all sets as the objects and the binary relations $R \subseteq A \times B$ as the morphisms from A to B, together with the relation composition as the composition and equality as the identity constitutes a category. This category is denoted by **Rel**.

Example 1.5. *(Discrete Categories)* A category $\mathcal{C}$ is called discrete if it only has the identity maps. Therefore, any set can be considered as a small discrete category.

$$
\begin{tikzcd}
A \arrow[loop above, "id_A"] & B \arrow[loop above, "id_B"] & C \arrow[loop above, "id_C"]
\end{tikzcd}
$$

Example 1.6. *(Some Finite Categories)* These are some finite categories:

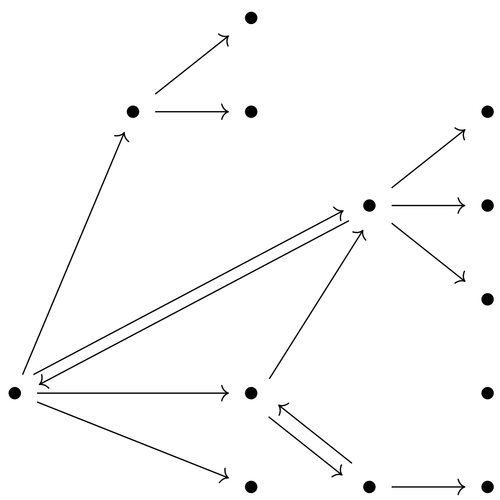
$$\mathbf{0}:$$

$$\mathbf{1} : \ \bullet$$

$$\mathbf{2} : \ \bullet \longrightarrow \bullet$$

$$\mathbf{3} : \ \bullet \longrightarrow \bullet \longrightarrow \bullet$$

Example 1.7. *(Preorders)* A small category $\mathcal{C}$ is called a preorder if for any two objects $A, B \in ob(\mathcal{C})$, the collection $Hom_{\mathcal{C}}(A, B)$ has at most one element. Spelling out the definition of a category in this special case, a preorder is actually a set, usually denoted by P with a binary relation $\leq \subseteq P \times P$ such that $x \leq x$, for any $x \in P$ and if $x \leq y$ and $y \leq z$ then $x \leq z$. There are many concrete examples of preorders. For instance, the set of integers $\mathbb{Z}$ with its usual order is a preorder. This set with the divisibility relation is another preorder. The prototype example of preorders is a set of subsets of some set X with inclusion as the order.



Remark 1.8. It is useful to think of preorders as the shadow of the usual categories, reducing all transformations between two objects to just one transformability between them. In the logical reading, this means that we collapse all the proofs between two statements to one provability map. Hence, in this sense logic can be considered as a special case of categories.

Example 1.9. *(Monoids)* A small category $\mathcal{C}$ is called a monoid if it has exactly one object. Spelling out the definition of a category in this special case, a monoid is actually a set, usually denoted by M with a binary operation

$\cdot : M \times M \to M$ and an element $e \in M$ such that $e \cdot x = x \cdot e = x$, for any $x \in M$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, for any $x, y, z \in M$. There are many concrete examples of monoids. For instance, the set of natural numbers $\mathbb{N}$ or $\mathbb{R}^+$ with their usual products are monoids. Moreover, any sets of endomaps of some set X that includes the identity and is also closed under composition is a monoid. This example is the prototype example of monoids.

Philosophical Note 1.10. A category is a combination of the two aforementioned extreme cases, a preorder and a monoid. The first handles the existence of different objects in a category and the second addresses different maps between any two objects.

Exercise 1.11. Check with all the details that all the previously claimed categories are actually categories.

Exercise 1.12. Show that the identity map of a given object is unique.

Now, it is reasonable to see the categorical formalization of some of the notions we talked about in the first session.

Example 1.13. *(Euclidean Geometry of the Plane)* The collection of all polygons P in $\mathbb{R}^2$ as the objects and $f_T : P \to Q$ as maps, where f_T is some formal map assigned to a distance preserving function $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T[P] = Q$, together with the usual composition and identity is a category.

Example 1.14. *(The Geometry of Maxwell's equations)* The collection of all the subsets U of the set of the lines going through the origin in $\mathbb{R}^5$ as objects and $f_T : U \to V$, where f_T is some formal map assigned to the function $T : \mathbb{R}^5 \to \mathbb{R}^5$ that preserves $[\mathbf{x}, \mathbf{y}] = x_0y_0 + x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4$ and $T[U] = V$, together with the usual composition and identity is a category.

Example 1.15. *(Vectors and tensors)* The collection $\{v\}_{v \in \mathbb{R}^n}$ as the objects and $A : v \to w$ as maps, where A is an $n \times n$ invertible matrix such that $Av = w$, together the usual composition and identity is a category. More generally, for any pair (p, q), the collection $\{T\}_{T \in \mathbb{R}^{n^{p+q}}}$ as the objects and $R : T \to S$ as maps, where R is an invertible $n \times n$ matrix R such that

$$S^{i'_1, \cdots, i'_p}_{j'_1, \cdots, j'_q} = \sum_{i_1, \cdots, i_p, j_1, \cdots, j_q} (R^{-1})^{i'_1}_{i_1} \cdots (R^{-1})^{i'_p}_{i_p} \, T^{i_1, \cdots, i_p}_{j_1, \cdots, j_q} \, R^{j_1}_{j'_1} R^{j_1}_{j'_1} \cdots R^{j_q}_{j'_q}.$$

together with the usual composition and identity is a category.

Philosophical Note 1.16. A category can be interpreted in two different ways. In its face, any category is just a structured graph interpretable as a syntactic algebraic theory describing the behaviour of some arrows. However, it is also possible to interpret it in a more semantical and geometrical manner. Here, there are two general approaches. The *petit* and the *gros* interpretations. In the first interpretation, we read the objects as an admissible family of models and maps as the structure preserving transformations. This covers the following more specific interpretations:

- *(Logical interpretation)* Objects as the statements and maps as the conditional proofs, i.e., the map $f : A \to B$ is a proof for B, using the assumption A,
- *(Bourbaki interpretation)* Objects as the structures of a given type and morphisms as the structure preserving transformations,
- *(Computer science interpretation)* Objects as the data types and morphisms as the computable transformations.

It is also possible to read the category itself as one huge model whose objects are the admissible parts of the model that are small enough to get observed and its maps are the admissible transformations between the parts. The following is a specific example of such interpretation:

- Objects as the points of a space and maps as the paths between them, i.e., a map $f : A \to B$ is interpreted as a path from A to B.
- Objects as the subspaces of a space and morphisms as the spatial maps between them.
- Objects as the linear subspaces of a linear space and morphisms as the linear maps between them.

Example 1.17. For instance, a monoid is just a syntactical entity consisting of a set together with a fixed element and a binary product satisfying some properties. The interpretation reads the one object of the category as a concrete set X, the morphisms as a set of concrete functions over X and the identity and composition as their usual concrete counterparts. In this sense, the interpretation tries to *realize* the abstract graph-like category by concrete notions.