Mathematical Structuralism, S03

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1 Category Theory (continued)

Here are two examples of the categories that admit both the petit and gros interpretations:

Example 1.1. Let X be an infinite set and $Fin(X)$ be the poset of all finite subsets of X with the inclusion as its partial order. As we have observed, any preorder including $(Fin(X), \subseteq)$ can be transformed to a category. Using the petit interpretation, this category will be read as the category of some sort of models, here the finite sets, while the gros interpretation reads it as the category of the finite approximations of the "huge" set X.

Example 1.2. Consider the category **FinVect**, constituting of $\mathbb{R}^n$, for any $n \in \mathbb{N}$, as the objects and the linear maps as the morphisms with the usual identity and composition. This category can be interpreted both as the category of all finite dimensional vector spaces (the models) or as the category of all finite dimensional approximations of an infinite dimensional vector space (the "huge" model).

Example 1.3. (*Variable Sets*) The collection of functions

$$\begin{array}{c} A_1 \\ {\scriptstyle f}\uparrow \\ A_0 \end{array}$$

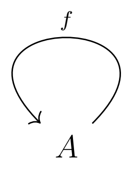
as the objects and the morphisms $\alpha : f \to g$ as the pair of functions (α_0, α_1), where $\alpha_0 : A_0 \to B_0$ and $\alpha_1 : A_1 \to B_1$ such that $\alpha_1 f = g\alpha_0$:

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & B_1 \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ A_0 & \xrightarrow{\alpha_0} & B_0 \end{array}$$

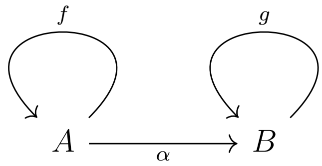
with the evident composition and identity, constitutes a category denoted by $\mathbf{Set}^{\to}$. Any object of the category can be interpreted as a *variable set*, varying over the discrete structure of time $\{0 \leq 1\}$. The set A_0 is the set of all the elements available at the moment $t = 0$ and the set A_1 is the set of the elements at the moment $t = 1$. Moving from $t = 0$ to $t = 1$, there are three main possibilities. Either some elements is created or some elements remain intact (up to some name change) or some of the distinct elements in A_0 become equal. These possible scenarios is formalized by a function f. The elements outside the range of f are the new elements in $t = 1$, while the elements in the range come from $t = 0$, with the latter two possible changes. Any map between these variable sets is naturally a pair of two maps, each for each moment of time, respecting the change of the sets through time.

Remark 1.4. In the previous example, there is nothing special about the structure $\{0, 1\}$ and it can be replaced by any other preorder or even by any other small category. Generalizing the variable sets in this way leads to very interesting conceptions of the incomplete sets growing over different structures of time. It also leads to some new models of the usual classical set theory. For that matter, it is enough to pick the variable sets and restrict ourselves to a subclass of complete ones. It is not easy to define these complete sets in one line. But to have an intuition, think about the variable sets so complete that in each moment of time, the set in that moment is large enough to have all the *imaginable elements* in that moment. For instance, the Cohen forcing to prove the independence of the axiom of choice is just the result of such a process: First setting a suitable structure to encode the growth of time and then letting the sets vary on that structure to finally harvest all the completed sets as a model of the usual classical set theory.

Example 1.5. (*Dynamical Systems*) The collection of functions

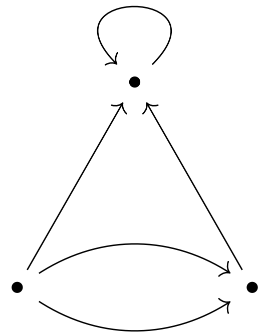


as the objects and the morphisms $\alpha : f \to g$ as a function $\alpha : A \to B$ such that $\alpha f = g\alpha$



with the evident composition and identity constitutes a category, denoted by $\mathbf{Set}^{\circlearrowleft}$. Any object of this category can be interpreted as a *dynamical system* consisting of a set A and a function $f : A \to A$, encoding the dynamism of the system. Of course, any map between the dynamic systems must be a function from the base sets preserving the dynamism.

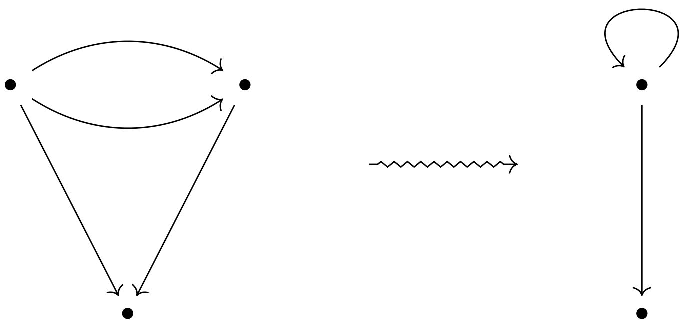
Example 1.6. (*Quivers*) Quivers are the directed multi-graphs as in the following figure:



formalized by

$$E \overset{s}{\underset{t}{\rightrightarrows}} V$$

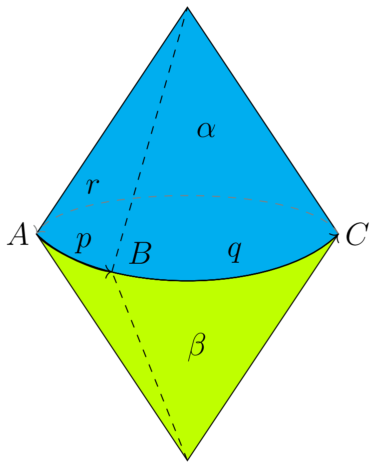
The set V is the set of vertices, the set E is the set of the edges and the two maps $s, t : E \to V$ are to encode the source and the target of any edge. The quiver morphisms then are the pairs of two functions mapping the vertices and the edges of the quivers, respecting the sources and the targets as in:



Formally, the quiver morphisms are the pairs of two functions $\alpha_V : V_0 \to V_1$ and $\alpha_E : E_0 \to E_1$ commuting with the source and the target functions, i.e., $\alpha_V s_0 = s_1 \alpha_E$ and $\alpha_V t_0 = t_1 \alpha_E$:

$$\begin{array}{ccc} E_0 & \overset{s_0}{\underset{t_0}{\rightrightarrows}} & V_0 \\ {\scriptstyle \alpha_E}\downarrow & & \downarrow{\scriptstyle \alpha_V} \\ E_1 & \overset{s_1}{\underset{t_1}{\rightrightarrows}} & V_1 \end{array}$$

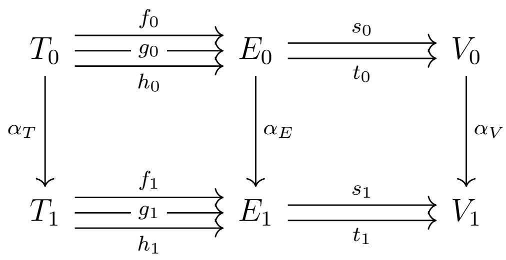
Example 1.7. (*2-quivers*) How to formalize the higher-order geometrical version of quivers as in the following figure?



It is easy to follow the formalization of the quivers again: A set V of the vertices, a set E of edges, and another set T of triangles with two maps $s, t : E \to V$ to encode the source and the target of any edge and three *face maps* $f, g, h : T \to E$, to record the different faces of a triangle.

$$T \overset{f}{\underset{h}{\xrightarrow{\;g\;}}} E \overset{s}{\underset{t}{\rightrightarrows}} V$$

In the figure, $V = \{A, B, C\}$, $E = \{f, g, h\}$, $T = \{\alpha, \beta\}$, canonical sources and targets and $f(\alpha) = f(\beta) = p$, $g(\alpha) = g(\beta) = q$ and $h(\alpha) = h(\beta) = r$. For morphisms, it is enough to have a triple $(\alpha_V, \alpha_E, \alpha_T)$ such that $\alpha_V : V_0 \to V_1$, $\alpha_E : E_0 \to E_1$ and $\alpha_T : T_0 \to T_1$ commuting with the source, the target and the face functions, i.e., the following diagram becomes commutative:



Leaving the many examples we had, we are ready to introduce the first categorical notion. We have seen that any map $f : A \to B$ can be interpreted as a transformation, changing the object A to the object B. Given this interpretation, one natural question is that when this transformation is reversible. Here is the categorical formulation:

Definition 1.8. A map $f : A \to B$ is called an isomorphism, if there exists a morphism $g : B \to A$ such that $fg = id_B$ and $gf = id_A$. This g is called *an* inverse of f.

Exercise 1.9. Prove that the inverse of a map is unique. Hence, it is well-defined to denote the inverse of f by f^{-1}.

Exercise 1.10. Prove that $id_A : A \to A$ is an isomorphism and if $f : A \to B$ and $g : B \to C$ are isomorphisms, then so is $g \circ f : A \to C$.

Exercise 1.11. Prove that in **Set**, the isomorphisms are the bijective maps. What are the isomorphisms in posets, monoids, $\mathbf{Set}^{\rightarrow}$ and $\mathbf{Set}^{\circlearrowleft}$?

Definition 1.12. *(Groupoids and Groups)* A groupoid is a category whose morphisms are all isomorphisms. A group is a groupoid with just one object. Spelling out the definition in this special case, a group is a monoid, usually denoted by G, such that for any $x \in G$, there exists $y \in G$ such that $x \cdot y = y \cdot x = e$.

Example 1.13. The category of all sets and bijective maps as morphisms with the usual composition and identity is a groupoid.

All the Examples **??**, **??**, **??** are groupoids.

Example 1.14. The prototype example of groups is a set of invertible functions over some set X that includes the identity and is closed under composition and inversion.

Philosophical Note 1.15. Groupoids can be interpreted as the formalizations of equality, where $f : A \to B$ is read as a proof or witness to show why A is equal to B. With this interpretation, it is easy to see that the group axioms are the natural conditions the reflexivity, the symmetry and the transitivity of the equality induce on the witnesses.

Definition 1.16. A function $f : G \to H$ is called a group homomorphism if it preserves the product. The category of groups and homomorphisms is denoted by **Grp**.

A digression: the representation theorems and the baby Erlangen program

1.0.1 Representation theorems

We have explained that how any category can be interpreted as the collection of the different ways that we can inspect a huge model. Is it possible to make this interpretation more formal? Let us begin with the two easy cases: the posets and the monoids. In theses case, we should ask if any poset is a poset of subsets of a concrete set and if any monoid is a monoid of concrete functions over a concrete set. The answer in both cases is positive.

Theorem 1.17. *(Cayley's Representation Theorem) Any monoid (group) is isomorphic to a monoid (group) of concrete functions over a concrete set.*

Proof. Let M be a monoid. Define the set X as the monoid itself and consider N as the set of all functions $f_m : X \to X$ defined by $f_m(x) = mx$, for $m \in M$. It is easy to see that $f_e = id$, since e is the left identity and $f_{mn} = f_m \circ f_n(x)$, since the product is associative. Hence, the map $F : M \to N$ defined by $F(m) = f_m$ is a homomorphism. By definition, F is clearly onto. It is also one to one, because if $F(m) = F(n)$, then $f_m = f_n$ which implies $f_m(e) = f_n(e)$. Hence, by the fact that e is also the right identity, we have $m = n$. For groups, note that if M is also a group, then $f_{m^{-1}} = f_m^{-1}$. Therefore, the set N is also a group. □

Theorem 1.18. *Any poset is isomorphic to a poset of subsets of a concrete set.*

Proof. Let $(P, \leq)$ be a poset. Set X as the set of all the subsets of P of the form $I_a = \{x \in P \mid x \leq a\}$. Define $F : P \to X$ by $F(a) = I_a$. Note that if $a \leq b$ then $F(a) \subseteq F(b)$, because if $x \leq a$, then $x \leq b$, by the transitivity of the order. F is clearly onto. It is also one to one, because if $F(a) = F(b)$, then $I_a = I_b$. By reflexivity, $a \leq a$. Hence, $a \in I_a = I_b$. Therefore, $a \leq b$. By a similar argument, $b \leq a$. Therefore, by anti-symmetry $a = b$. This means the inverse function $G : X \to P$ sending I_a to a is well-defined. To show that G preserves the order, we have to show that if $F(a) \subseteq F(b)$, then $a \leq b$. □

Remark 1.19. It is worth mentioning that the previous theorems need and also use all the conditions in the definition of a monoid and a poset, respectively. Therefore, they imply that the conditions are necessary and sufficient to capture the abstract behaviour of a family of functions over a set, including the identity and being closed under composition and a set of subsets of a given set, respectively.

As the next natural step, we generalize the previous two cases to any category:

Theorem 1.20. *Any small category is "isomorphic" to a category of concrete sets with concrete functions.*

Proof. Let $\mathcal{C}$ be category. To any object A of $\mathcal{C}$ assign the set $A_* = \{g : B \to A \mid g \in Morph(\mathcal{C})\}$ and to any map $f_* : A \to B$, the function $f_* : A_* \to B_*$ defined by $f_*(g) = fg$. Now consider the category $\mathcal{D}$ consisting of A_* as objects and $f_* : A_* \to B_*$ as morphisms. Then, defining $F : \mathcal{C} \to \mathcal{D}$ by sending A to A_* and $f : A \to B$ to $f_* : A_* \to B_*$ we can reach an isomorphism. It is easy to see that F preserves composition and identity. F is also one-to-one on objects and morphisms. For objects the claim is obvious. For morphisms, if $f, g : A \to B$ and $f_* = g_* : A_* \to B_*$, then since $id_A \in A_*$ we have $f_*(id_A) = g_*(id_A)$, which implies $f = g$. Now, it is easy to define the converse of F and check that it the respects identity and the composition. □

Now, it is natural to extend the previous representation theorems to all categories to see if it is possible to represent any category as a category of sets together with some concrete functions as morphisms? This time the answer is negative and its proof is beyond the scope of this section. However, it is worth mentioning that this negative result seriously affects the universal applicability of Bourbaki's set-based approach to structures.

1.0.2 Baby Erlangen program

So far, we have seen that any monoid (group) is actually a monoid (group) of concrete functions over a concrete set. Therefore, any group is a group of transformations over some set. Now, following Klein's Erlangen program, it is reasonable to ask that given the group of transformations, what different geometries it may be possible.

Definition 1.21. Let X be a set and $Aut(X)$ be the group of all permutations of X, i.e., the bijections from X to itself. A homomorphism from $F : G \to Aut(X)$ is called an action of G on X. Sometimes, for simplicity, we write gx for $F(g)(x)$. Two actions $F : G \to Aut(X)$ and $F' : G \to Aut(Y)$ are called isomorphic if there exists a bijection $\phi : X \to Y$ such that $F(g)\phi = \phi F'(g)$, for any $g \in G$.

Example 1.22. The trivial example of an action of the group G is the action of G on itself, defined by $F : G \to Aut(G)$, where $F(g) = f_g$ and $f_g(x) = gx$. For a more sophisticated example, let us do the trivial example in a modular manner. Let N be a subset of G closed under some operations that we meet later. Then, we call two elements $f, g \in G$ congruent modulo N if $f^{-1}g \in N$. It is reasonable to expect that the congruence to be an equivalence relation and if we denote the set of the equivalence classes by G/N, the function $G \to Aut(G/N)$ defined by $g[h] = [gh]$ becomes an action. These expectations are not automatically true. To make them true, N must be closed under product, inverse and all the operations in the form $x \mapsto g^{-1}(x)g$, for any $g \in G$.