Mathematical Structuralism, S04

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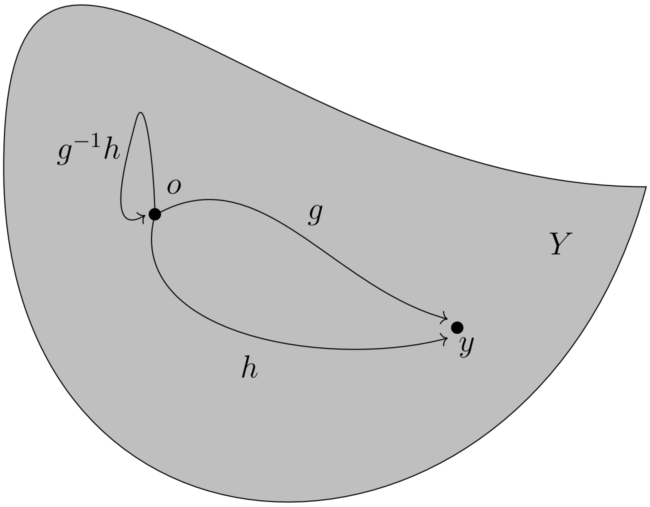
1 Category Theory

A digression: the representation theorems and the baby Erlangen program

Baby Erlangen program

Example 1.1. The trivial example of an action of the group G is the action of G on itself, defined by $F : G \to Aut(G)$, where $F(g) = f_g$ and $f_g(x) = gx$. For a more sophisticated example, let us do the trivial example in a modular manner. Let N be a subset of G closed under some operations that we meet later. Then, we call two elements $f, g \in G$ congruent modulo N if $f^{-1}g \in N$. It is reasonable to expect that the congruence to be an equivalence relation and if we denote the set of the equivalence classes by G/N, the function $G \to Aut(G/N)$ defined by $g[h] = [gh]$ becomes an action. These expectations are not automatically true. To make them true, N must be closed under product, inverse and all the operations in the form $x \mapsto g^{-1}xg$, for any $g \in G$.

In group theory literature, there is a characterization theorem, stating that any G-action *is* the "disjoint union" of the G-actions introduced in Example 1.1. We will repeat the usual argument here. Let $F : G \to Aut(X)$ be a G-action. Define the reachability relation R on X by $(x, y) \in R$, if there exists $g \in G$ such that $gx = y$. It is not hard to prove that the relation R is an equivalence relation, using the fact that G is actually a group. Each equivalence class inherits a G-action from the original G-action F. The reason simply is that if x is an element in the class and $g \in G$, the result of the action, namely gx, is in the same class as x. Finally, we will show that each of these restricted G-actions on the equivalence classes is isomorphic to a G-action of the type introduced in the Example 1.1. Let Y be one of these classes. Set an arbitrary element $o \in Y$:



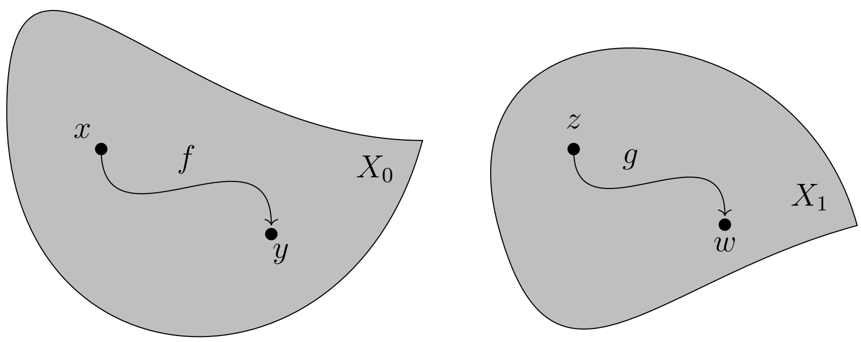
Define $N = \{g \in G \mid go = o\}$. It is easy to see that N has the required closure properties, namely the closure under product, inverse and the operations $x \mapsto g^{-1}xg$, for any $g \in G$. Define $\phi : G/N \to Y$ by $\phi([g]) = go$. The function is well-defined and one-to-one, because $\phi([g]) = \phi([h])$ iff $go = ho$ iff $g^{-1}ho = o$ iff $g^{-1}h \in N$ iff $[g] = [h]$. It is not hard to see that ϕ is an isomorphism between the G-actions. The important thing is that the function is surjective, because any y in the class is reachable from o and hence $go = y$, for some $g \in G$.

Remark 1.2. Note that the above construction has some unsatisfactory elements. First, some of its parts are chosen in a non-canonical manner, like the element $o \in Y$. These choices do not affect the construction, but makes the construction tricky at best and resistant to generalizations at worst. Secondly, although the notion of action is equivalently meaningful for monoids, the above construction seriously uses the fact that G is a group and hence it does not suggest any way to handle the monoid case, as well.

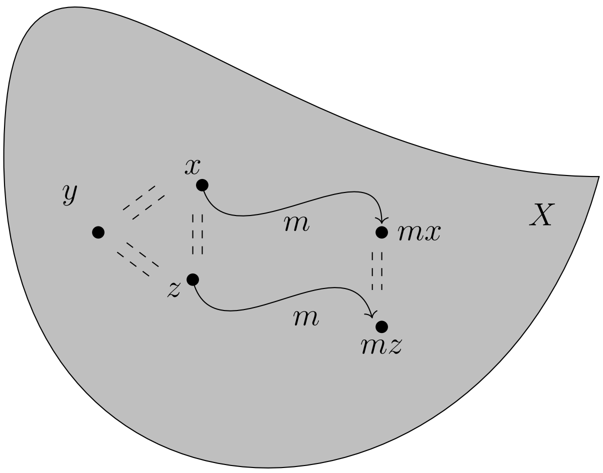
To overcome the issues mentioned above, let us provide a characterization theorem again. This time, we use the canonical approach consisting of simple, intuitive and justifiable steps that uses no ingredient except what it is essentially required. As a result, this time, everything is more transparent so much so that we can even address the case of monoids.

Definition 1.3. Let X be a set and $End(X)$ be the monoid of all functions on X. A homomorphism $F : M \to End(X)$ is called an action of M on X or an M-action, for short. Sometimes, we write mx for $F(m)(x)$, for simplicity. Two M-actions $F : M \to End(X)$ and $F' : M \to End(Y)$ are called isomorphic, if there exists a bijection $\phi : X \to Y$ such that $F(m)\phi = \phi F'(m)$, for any $m \in M$.

The trivial example of an M-action is the action of M on itself, defined by $F : M \to End(M)$, where $F(m) = f_m$ and $f_m(x) = mx$. To provide a characterization theorem, we will introduce two methods to construct the new M-actions from the old. First, the "*disjoint union*". Let $\{F_i : M \to End(X_i)\}_{i \in I}$ be a family of M-actions. Define $X = \sum_{i \in I} X_i = \{(i, x) \mid i \in I, x \in X_i\}$ with the fibrewise M-action $m(i, x) = (i, mx)$. This is clearly an M-action:



The second method, the "*quotient*" operation, picks one M-action and glue some of its elements together to get a new one. More precisely, let $F : M \to End(X)$ be an M-action and $R \subseteq X \times X$ be a set of the pairs of the elements of X that we want to glue to each other. It is possible to provide the minimal M-action in which these intended equalities are *forced* to hold. It is enough to define the equivalence relation $\sim$ as the least equivalence E, extending the relation R and respecting the M-action, i.e., if $(x, y) \in E$ then $(mx, my) \in E$, for any $m \in M$. (Why does such an equivalence relation exist?) Then, define Y as the set of the equivalence classes with respect to $\sim$ and define $m[x] = [mx]$. (Why is it well-defined, i.e., independent of the representative of the classes?)



To prove that any action is constructible from the basic action via disjoint union and quotient operations, let $F : M \to End(X)$ be an arbitrary M-action. Then, define Z as the quotient of the disjoint union $Y = \sum_{x \in X} M$ by the set $\{((x, m), (y, n)) \in Y^2 \mid mx = ny\}$ and define $\phi : Z \to X$ by $\phi[(x, m)] = mx$. It is clearly well-defined and one-to-one. It is also surjective since $\phi([(x, e)]) = x$. It will be easy to define the converse function and show that it is an M-action.

Now, again, it is a natural question that if it is possible to generalize the aforementioned characterization to any small category. The answer is again positive. But we first need the right notions of an action (realization) and isomorphism between these actions (realizations), for categories. The first is called a *functor* and the second is *natural isomorphism.* We will spend some time on these notions to set the scene to provide a characterization theorem for the small categories.