Mathematical Structuralism, S05

Amir Tabatabai

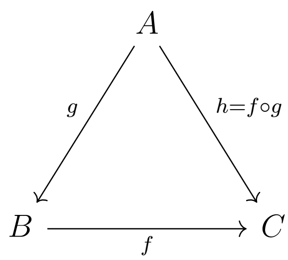
December 31, 2020

1 Category Theory (continued)

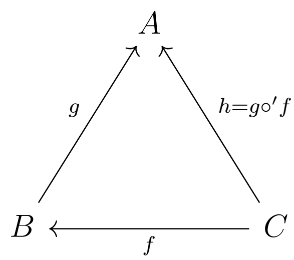
1.1 New categories from the old

Example 1.1. (*Opposite Category*) Let $\mathcal{C}$ be a category. By its dual (opposite), $\mathcal{C}^{op}$, we mean a category with same collection of objects and morphisms as of $\mathcal{C}$ with the source and the target assignment swapped and $f \circ' g = g \circ f$:

$\mathcal{C}$:

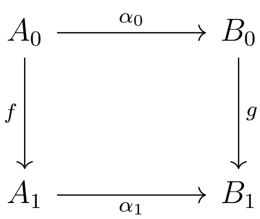


$\mathcal{C}^{op}$:

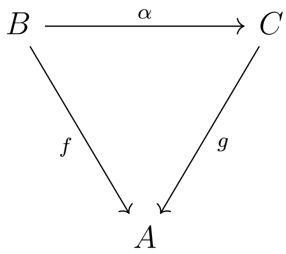


Exercise 1.2. Show that if $f : A \to B$ is an isomorphism in $\mathcal{C}$, then it is also an isomorphism in $\mathcal{C}^{op}$. Use this fact to show that the dual of a groupoid is also a groupoid.

Example 1.3. (*Arrow Category*) Let $\mathcal{C}$ be a category. Then, by the arrow category $\mathcal{C}^{\to}$, we mean the category with the morphisms of $\mathcal{C}$ as the objects and the pair of morphisms $(\alpha_0, \alpha_1) : f \to g$ of $\mathcal{C}$ as the morphism, where $\alpha_0 : A_0 \to B_0$ and $\alpha_1 : A_1 \to B_1$ such that $g\alpha_0 = \alpha_1 f$:

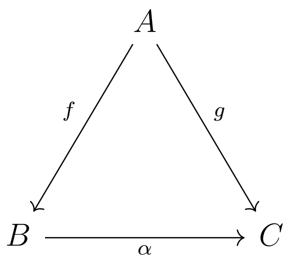


Example 1.4. (*Slice Category*) Let $\mathcal{C}$ be a category and A be an object. Then, by the *slice* category or *over* category $\mathcal{C}/A$, we mean the category with the morphisms $f : B \to A$ of $\mathcal{C}$ with the target A as the objects and $\alpha : f \to g$ as the morphism, where $\alpha : B \to C$ is a morphism in $\mathcal{C}$ such that $g\alpha = f$:



As a concrete example, note that in any poset $(P, \leq)$, the slice P/a is just P restricted to the elements less than or equal to a. As another example, observe that $\mathbf{Set}/\{0,1\}$ is actually the category of partitioned sets into two parts, i.e., (A, A_0, A_1), where $A = A_0 \cup A_1$ and $A_0 \cap A_1 = \emptyset$ and functions, i.e., $f : (A, A_0, A_1) \to (B, B_0, B_1)$, where $f : A \to B$ is a function and $f[A_i] \subseteq B_i$, for any $i \in \{0,1\}$. The reason simply is that any map $m : A \to \{0,1\}$ is nothing but the partition of A into $m^{-1}(0)$ and $m^{-1}(1)$ and any commutative triangle means respecting these parts.

Example 1.5. (*Coslice Category*) Let $\mathcal{C}$ be a category and A be an object. Then, by the *coslice* category or *under* $A/\mathcal{C}$, we mean the category with the morphisms $f : A \to B$ of $\mathcal{C}$ with the source A as the objects and $\alpha : f \to g$ as the morphism, where $\alpha : B \to C$ is a morphism in $\mathcal{C}$ such that $g = \alpha f$:



As a concrete example, note that in any poset $(P, \leq)$, the coslice a/P is just P restricted to the elements greater than or equal to a. As another example, observe that $\{0\}/\mathbf{Set}$ is actually the category of pointed sets, i.e., (A, a), where $a \in A$ and pointed functions, i.e., $f : (A, a) \to (B, b)$, where $f : A \to B$ is a function and $f(a) = b$. The reason simply is that any map $\{0\} \to A$ is nothing but the choice of an element and any commutative triangle means respecting this chosen element.

Example 1.6. (*Product of Categories*) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Then by $\mathcal{C} \times \mathcal{D}$, we mean the category with the pairs (C, D) as the objects, where C and D are the objects of $\mathcal{C}$ and $\mathcal{D}$, respectively and the pair $(f, g) : (C, D) \to (E, F)$ as the morphisms, where $f : C \to E$ and $g : D \to F$ are morphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively:

$$(C, D) \xrightarrow{(f,g)} (E, F)$$

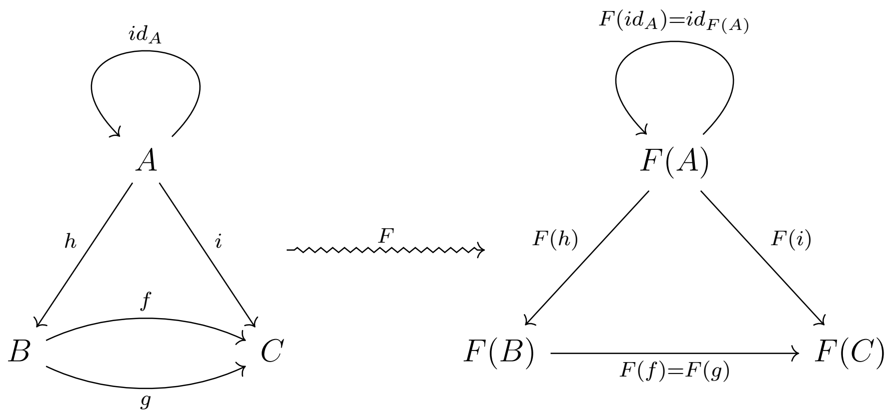
Note that this construction generalizes the product of monoids and groups on the one hand and the product of sets on the other.

Example 1.7. (*Coproduct of Categories*) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Then by $\mathcal{C} + \mathcal{D}$, we mean the category with $(0, C)$ and $(1, D)$ as the objects, where C and D are the objects of $\mathcal{C}$ and $\mathcal{D}$, respectively and $(0, f) : (0, C) \to (0, C')$ and $(1, g) : (1, D) \to (1, D')$ as the morphisms, where $f : C \to C'$ and $g : D \to D'$ are morphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively. Note that this construction generalizes the disjoint union of sets.

1.2 Functors and Natural Transformations

To find the natural formalization of realizations for categories, note that a realization of a monoid (or a group) is an assignment that maps the only abstract object of the monoid (or the group) to a concrete set and any abstract morphism (i.e., the elements of the monoid) to a concrete function on the set, respecting the identity and the composition:

Definition 1.8. (*Functors*) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. By a functor $F : \mathcal{C} \to \mathcal{D}$, we mean a pair of two assignments F_0 and F_1, such that F_0 maps any object A of $\mathcal{C}$ to an object of $\mathcal{D}$, denoted by $F_0(A)$ and F_1 maps any morphism f of $\mathcal{C}$ to a morphism in $\mathcal{D}$, denoted by $F_1(f)$, respecting the source, the target, the identity and the composition operations:



Usually, for simplicity, one drops the subscript in F_0 and F_1 and denote both by F.

Philosophical Note 1.9. It is possible to interpret a functor $F : \mathcal{C} \to \mathcal{D}$ as a way to interpret the discourse $\mathcal{C}$ in the discourse $\mathcal{D}$, as a way to realize $\mathcal{C}$ in $\mathcal{D}$, as a $\mathcal{C}$-indexed family in $\mathcal{D}$ or a $\mathcal{C}$-variable object in $\mathcal{D}$. For instance, any M-action $F : M \to End(X)$ is a functor $M \to \mathbf{Set}$, realizing the only abstract object of M by X and the morphisms of M by real functions over X, according to F. As another example, it is possible to see that any variable set in $\mathbf{Set}^{\to}$ is actually a functor from $\mathbf{2}$ to $\mathbf{Set}$, realizing the abstract graph

$$\mathbf{2} : \bullet \longrightarrow \bullet$$

by the concrete sets and functions. Similarly, any quiver is a functor from the category

$$\bullet \rightrightarrows \bullet$$

to $\mathbf{Set}$, realizing the abstract points as concrete sets of vertices and edges and abstract arrows as concrete source and target functions. For an example of the other interpretation, we have already seen that any object in $\mathbf{Set}^{\to}$ can be read as a variable set over the structure of time, encoded by the category $\mathbf{2}$.

Example 1.10. Any homomorphism between two monoids is a functor. Any order-preserving map between two posets is a functor. It is worth mentioning that functors are the right common generalization of composition- and order-preserving maps.

Example 1.11. The assignment mapping any set A to its powerset $P(A)$ and any function $f : A \to B$ to the function $P(f) : P(A) \to P(B)$, defined by $P(f)(S) = f[S] = \{f(a) \mid a \in S\}$ is a functor from $\mathbf{Set}$ to itself. Similarly, the functor $\bar{P} : \mathbf{Set} \to \mathbf{Set}^{op}$, mapping any set A to its powerset $P(A)$ and any function $f : A \to B$ to the function $\bar{P}(f) : P(B) \to P(A)$, defined by $\bar{P}(f)(S) = f^{-1}(S) = \{a \in A \mid f(a) \in S\}$ is a functor.

Example 1.12. The assignment mapping any object (A, B) in $\mathbf{Set} \times \mathbf{Set}$ to $A \times B$ and any morphism $(f, g) : (A, B) \to (C, D)$ of $\mathbf{Set} \times \mathbf{Set}$ to the function $f \times g : A \times B \to C \times D$ defined by $[f \times g](a, b) = (f(a), g(b))$ is a functor.

Example 1.13. The assignment mapping any object (A, B) in $\mathbf{Set} \times \mathbf{Set}$ to $A + B = \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}$ and any morphism $(f, g) : (A, B) \to (C, D)$ of $\mathbf{Set} \times \mathbf{Set}$ to the function $f + g : A + B \to C + D$ defined by $[f + g](0, a) = (0, f(a))$ and $[f + g](1, b) = (1, g(b))$ is a functor.