Mathematical Structuralism, S07

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January 15, 2021

1 Category Theory (continued)

1.1 Functors and Natural Transformations

Example 1.1. *(Fundamental set Π_0)* Let **Quiv** be the category of quivers (directed multi-graphs). For any quiver Q, define the equivalence relation $\sim$ on $V(Q)$ by $v \sim w$ iff there exist two paths of edges in $E(Q)$ (including the paths with length zero), one starting from v and ending in w and one starting from w and ending in v. (Why is it an equivalence relation?) Define the assignment $\Pi_0 : \mathbf{Quiv} \to \mathbf{Set}$ on objects by $\Pi_0(Q)$ as the set of equivalence classes in $V(Q)$ and on quiver morphism $f : Q \to Q'$ by $\Pi_0(f)([v]) = [f(v)]$. (Why is it well-defined?) The assignment Π_0 is a functor. It measures how connected the quiver is. It is also possible to use a more refined version in which the functor returns not only the set $\Pi_0(Q)$ but also its underlying order, defined by $[v] \leq [w]$ iff there exists a path from v to w. (Why is it a well-define poset order?) It is not hard to see that $\Pi_0(f)$ also respects this order. Denote this functor by $\Pi_0^d : \mathbf{Quiv} \to \mathbf{Poset}$.

Remark 1.2. Note that Π_0 is not faithful as it sends any two connected quivers to a singleton. The same also holds for Π_0^d.

Exercise 1.3. Prove that functors preserve isomorphisms, i.e., if $F : \mathcal{C} \to \mathcal{D}$ is a functor and $f : A \to B$ is an isomorphism in $\mathcal{C}$, then $F(f) : F(A) \to F(B)$ is an isomorphism in $\mathcal{D}$.

Definition 1.4. A functor $F : \mathcal{C} \to \mathcal{D}$ is called faithful if for any $f \neq g : A \to B$, we have $F(f) \neq F(g) : F(A) \to F(B)$. In other words, F is faithful if $F : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(A), F(B))$ is one-to-one. It is called full if any $h : F(A) \to F(B)$ is equal to $F(f)$ for some $f : A \to B$. In other words, F is full if $F : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(A), F(B))$ is surjective.

Example 1.5. An order-preserving map $f : (P, \leq_P) \to (Q, \leq_Q)$ between two posets is always faithful. It is full iff it is an order-embedding, i.e., $a \leq_P b$ iff $f(a) \leq_Q f(b)$. A homomorphism between two monoids is faithful iff it is one-to-one and it is full iff it is surjective. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ is faithful but not full.

Exercise 1.6. The product functor $(-) \times (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ is faithful but not full.

Exercise 1.7. Let $f : A \to B$ be a morphism in a category $\mathcal{C}$. When is $f_* : \mathcal{C}/A \to \mathcal{C}/B$ faithful? When is it full?

Exercise 1.8. Let $\mathcal{C}$ be a locally small category. Is $Hom_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ full or faithful?

Philosophical Note 1.9. *Non-faithful functors* provide some room to simplify the original object A in a discourse $\mathcal{C}$ to a simpler object $F(A)$ in $\mathcal{D}$. When $F(A)$ is "computable" in a relatively easy way, F can be useful in showing that two given objects in $\mathcal{C}$ are not isomorphic. The strategy is as follows: Assume that an isomorphism $f : A \to B$ exists between two given objects A and B. Then, by the application of the functor F, we must have an isomorphism between $F(A)$ and $F(B)$ in $\mathcal{D}$. Now, compute both $F(A)$ and $F(B)$ and show that they can not be isomorphic. The basic version of this argument is when we find an "easy-to-check" property P such that it is invariant under the given isomorphisms and A and B disagree on this property P. For instance, to prove that the two groups $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are not isomorphic, it is enough to observe that the latter has the property $P = \forall x \exists y (y = x + x)$, while the former lacks it. Note also that P is a group-theoretic property, meaning it is invariant under all group isomorphisms. This argument is a special kind of the argument above, using a groupoid $\mathcal{C}$ of objects together with their isomorphisms and the functor $P : \mathcal{C} \to \{0, 1\}$ to capture the invariant-under-isomorphism property P, where $\{0, 1\}$ is a discrete category encoding true and false values.
It is also possible to have more complex examples, using more sophisticated categories for $\mathcal{D}$. For instance, consider the following quivers:

$$Q : \quad \begin{tikzcd} \bullet \arrow[d] \arrow[dr, bend right=20] \arrow[dr, bend left=30] & \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad Q' : \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \end{tikzcd}$$

They are not isomorphic, since the forgetful functor V : **Quiv** $\to$ **Set** maps Q to a three element set (the set of vertices) and Q' to a four element set. These two sets can not be isomorphic in **Set**. Hence, Q and Q' are not isomorphic as quivers. Note that the functor V is easy to compute and this is the key element that makes it useful here. Moreover, it is important to observe that showing two sets are not isomorphic boils down to an easy cardinality argument. However, as the functor is not faithful, it has its own blind spots. For instance, in the following situation

$$P: \quad \begin{tikzcd} \bullet \arrow[d] \arrow[dr] & \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad P': \quad \begin{tikzcd} \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet \arrow[ul] \end{tikzcd}$$

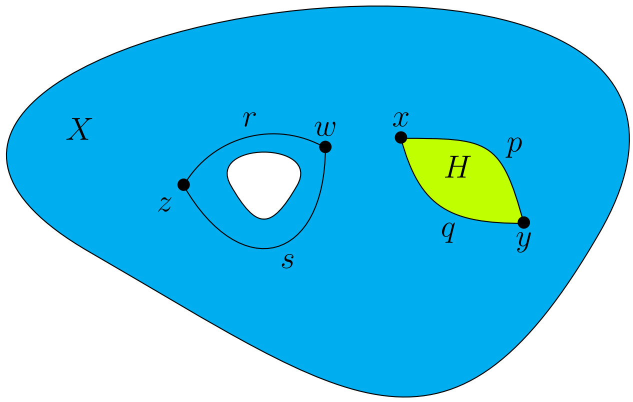
both functors V and E are blind to the difference. In such cases, it is reasonable to use more sophisticated functors. But, remember, they must remain relatively easier to handle than the original object. In this case, we use the functor Π_0. Since, $\Pi_0(P)$ is a three element set while $\Pi_0(P')$ is just a singleton, P and P' are not isomorphic as quivers. As the last example, consider the following two quivers:

$$R: \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, bend left] \arrow[dr, bend right] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad R': \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet \arrow[r] \arrow[ur, bend left] \arrow[ur, bend right] & \bullet \arrow[u] \end{tikzcd}$$

Here, all the three functors V, E and Π_0 agree. However, $\Pi_0^d(R)$ is a lozenge while $\Pi_0^d(R')$ is just a line.
As another example, consider the category $\mathcal{R}$ of Example **??**. To show that $\mathbb{Q}$ and $\mathbb{R}$ are not isomorphic in $\mathcal{R}$, it is enough to consider the forgetful functor $F : \mathcal{R} \to$ **Set**, since $F(\mathbb{Q})$ is countable, while $F(\mathbb{R})$ is uncountable and they can not be isomorphic as sets. However, to show that $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic in $\mathcal{R}$, the forgetful functor does not work, as the underlying sets have equal cardinality. In this case, it is useful to have the more refined functor V_I, for $I(x) = \{x^2 + 1 = 0\}$. Here, we have $V_I(\mathbb{R}) = \emptyset$, while $V_I(\mathbb{C}) = \{i, -i\}$ and these two sets are not isomorphic.

Example 1.10. (*Fundamental Groupoid* Π_1) Let **Top** be the category of all topological spaces with continuous functions. For any topological space

X, consider the set of paths in X from x to y, i.e., all continuous functions $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$, denoted by $Path_X(x,y)$. First, note that it is again possible to define the functor $\Pi_0 : \mathbf{Top} \to \mathbf{Set}$ by setting $\Pi_0(X)$ as X up to the equivalence $\sim$ defined by $x \sim y$ if there exists a path in X from x to y. The function $\Pi_0(f)$ is also defined canonically as before. The functor Π_0 measures how connected the space X can be. Now, to define another functor, lift these considerations one level up, i.e., define the equivalence relation $\sim$ on $Path_X(x,y)$ by $p \sim q$ iff there exists a surface in X filling between p and q, i.e., a continuous function $H : [0,1] \times [0,1] \to X$ such that H maps $\{0\} \times [0,1]$ to x, $\{1\} \times [0,1]$ to y and the restrictions of H to $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ becomes p and q, respectively. (Why is it an equivalence relation?)



In the figure, the image of H is depicted by the green area and hence $p \sim q$, while r and s can not be in the same class as the white hole in the middle prevents any surface between r and s. Now, define $\Pi_0(X)$ as the groupoid with the objects as the elements of X, the morphisms from x to y as $Path_X(x,y)$ and composition and identity as the canonical pasting paths to each other and the class of the constant path. (Why is composition well-defined? Why is the constant map the identity morphism?) Define the assignment $\Pi_1 : \mathbf{Top} \to \mathbf{Groupoid}$ on objects by $\Pi_1(X)$ and on a morphism $f : X \to Y$ by the functor $\Pi_1(f)$ defined by $\Pi_1(f)(x) = f(x)$ and $\Pi_1(f)([p]) = f[p]$. (Why is it well-defined?) The assignment Π_1 is a functor. It is possible to simplify the functor Π_1 with some non-canonical choice for a base point. Let X be a space and $x \in X$ be a point in X. Now, restrict the groupoid $\Pi_(X)$ to the object x and the morphisms over x. This is also a functor, usually denoted by π_1, this time from the category of pointed spaces, denoted by $\mathbf{Top}_*$ to the category $\mathbf{Grp}$. Both Π_1 and π_1 measure the 2-holes in a space X as Π_0 measured 1-holes. (1-hole means disconnectedness. Right?)

For instance, for the space $B_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and any possible choice for the base point $a \in B_2$, the group $\pi_1(B_2, a)$ is just a singleton, as any path over a in B_2 can be filled and B_2 (why?) or in other words as B_2 has no holes. At the same time, for the circle $S_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, the group $\pi_1(S_1, a)$ for any base point $a \in S_1$ is $\mathbb{Z}$, as any path over a in S_1 is uniquely determined by the number it goes around S_1. (Why?) These are obvious claims. But intuitively, they are just clear.

Philosophical Note 1.11. It is possible to interpret any topological space X as a set with multiplicities, any path $p : x \to y$ as a *proof* of equality between x and y, any surface between two paths $p, q : x \to y$ as a *proof* of equality between p and q and so on. With this interpretation, while $\Pi_0(X)$ computes the set of distinct elements of X, the functor $\Pi_1(X)$ computes the distinct proofs between two equal elements.