Mathematical Structuralism, S08

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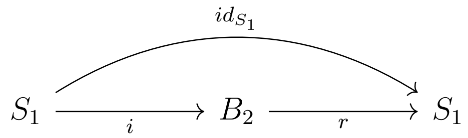
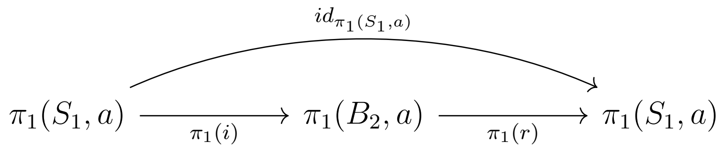
1 Category Theory (continued)

1.1 Functors and Natural Transformations

Example 1.1. (*Application of the Fundamental Groups*) We want to prove *Brouwer's fixed point theorem* for 2-ball $B_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ that states that any continuous function $f : B_2 \to B_2$ has a fixed point. For the sake of contradiction, assume f does not have a fixed point. Then, for any $(x, y) \in B_2$, we have $f(x, y) \neq (x, y)$. Define $r : B_2 \to S_1$ in the following way: Take the directed line L, connecting $f(x, y)$ to (x, y) and define $r(x, y)$ as the intersection of L and the border of B_2 which is S_1. By definition, the restriction r to S_1 is the identity function. Therefore, if we denote the inclusion of S_1 in B_2 by $i : S_1 \to B_2$, we have:



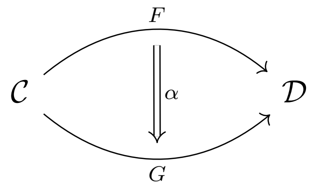
Since $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ is a functor, if we pick an arbitrary $a \in S_1$, we have:



which is impossible, as $\pi_1(S_1, a)$ is isomorphic to $\mathbb{Z}$, while $\pi_1(B_2, a)$ is a singleton group.

Remark 1.2. In almost all the applications of the fuctors we have seen so far, except maybe the previous example, the only thing we used was the fact that the functors from one discourse to the other preserve isomorphisms, as they are expected to preserve the corresponding notion of "sameness". Following such observations, one may find it tempting to restrict category theory to groupoids as the formalization of a discourse equipped with its notion of sameness. The previous example is just a simple instance to show that this temptation is somewhat naive. Morphisms and not just isomorphisms are important to capture the behavior of an object and it is useful if we know how to transfer them from one discourse to another.

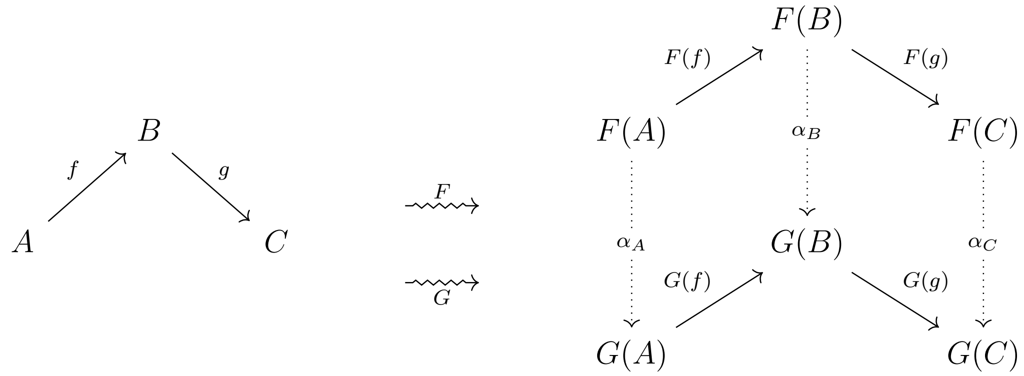
Definition 1.3. (*Natural Transformations*) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. By a natural transformation $\alpha : F \Rightarrow G$, depicted as



we mean an assignment mapping any object of $\mathcal{C}$ to a morphism $\alpha_C : F(C) \to G(C)$ in $\mathcal{D}$ such that for any morphism $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\alpha_B]{} & G(B) \end{array}$$

Philosophical Note 1.4. If we read functors $F, G : \mathcal{C} \to \mathcal{D}$ as two $\mathcal{C}$-variable objects in $\mathcal{D}$, then any natural transformation $\alpha : F \Rightarrow G$ is a transformation between these variable objects. Naturally, any transformation between variable objects must specify the way we change the object $F(C)$ to the object $G(C)$ in $\mathcal{D}$, for each parameter $C \in ob(\mathcal{C})$. These changes can not be arbitrary. They must respect the changes in parameter in $\mathcal{C}$:



Philosophical Note 1.5. Let us read two functors $F, G : \mathcal{C} \to \mathcal{D}$ as two *construction methods* that read an object in $\mathcal{C}$ and transform it to an object in $\mathcal{D}$. When can we call F and G "equal" as two methods of construction? Of course we do not want to restrict ourselves to the very strict equality that demands the functors to be equal both on the objects and the morphisms. This is just too restrictive. For instance, consider $F, G : \mathbf{Set} \to \mathbf{Set}$ as $F(A) = A \times \{0\}$ and $G(A) = A \times \{1\}$. In this case, although F and G are not strictly equal, they must be considered as the same methods of construction, as they are only different up to an isomorphism. Using this criterion, one natural candidate for the intended equality between F and G is the existence of an isomorphism between $F(A)$ and $G(A)$, for any object A in $\mathcal{C}$. However, it is clear that any random assignment of isomorphisms between $F(A)$ and $G(A)$ does not work. The isomorphisms must be assigned in a *uniform* way, as we want F and G to be equal as two methods of constructions not two mere structureless assignments. This uniformity demands the isomorphisms to be somewhat independent of the choice of the object A. Of course, one may object that the isomorphisms clearly depend on the object A (the source and the target of the isomorphism, for instance), but at same time it is intuitively meaningful to talk about the constructions that apply the *same* method to *different* objects. An example may be more illuminating. Consider the canonical isomorphism $s_{A,B} : A \times B \to B \times A$ defined by $s_{A,B}(a, b) = (b, a)$ that shows the order in the product of two sets is not important. This map clearly depends on the choice of A and B, but at the same time it is defined in a uniform way of "swapping the elements in a pair" which dos not use the sets in an essential way. Natural transformations is historically developed for the sole purpose of capturing this very intuition of uniformity.

Example 1.6. The assignment $s : id_{\mathbf{Set}} \Rightarrow P$ defined by $s_A : A \to P(A)$ as $s_A(a) = \{a\}$ is a natural transformation. It is natural simply because if $f : A \to B$ maps $a \in A$ to $f(a) \in B$, then $P(f)$ maps $\{a\}$ to $f[\{a\}] = \{f(a)\}$.

$$
\begin{array}{ccc}
A & \xrightarrow{\{-\}} & P(A) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f[-]} \\
B & \xrightarrow[\{-\}]{} & P(B)
\end{array}
\qquad\qquad
\begin{array}{ccc}
a & \xmapsto{\{-\}} & \{a\} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f[-]} \\
f(a) & \xmapsto[\{-\}]{} & \{f(a)\}
\end{array}
$$

Example 1.7. The assignment $i : id_{\mathbf{Set}} \Rightarrow (P^{\circ})^{\circ}$ defined by $i_A : A \to PP(A)$ as $i_A(a) = \{S \subseteq A \mid a \in S\}$ is a natural transformation:

$$
\begin{array}{ccc}
A & \xrightarrow{i_A} & P(P(A)) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle P^{\circ}(P^{\circ}(f))} \\
B & \xrightarrow[i_B]{} & P(P(B))
\end{array}
\qquad
\begin{array}{ccc}
a & \xmapsto{i_A} & \{S \subseteq A \mid a \in S\} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle P^{\circ}(P^{\circ}(f))} \\
f(a) & \xmapsto[i_B]{} & \{T \subseteq B \mid f(a) \in T\}
\end{array}
$$

Note that $P^{\circ}(P^{\circ}(f))(\mathcal{S}) = (P^{\circ}(f))^{-1}(\mathcal{S}) = \{T \subseteq B \mid f^{-1}(T) \in \mathcal{S}\}$ which maps $\{S \subseteq A \mid a \in S\}$ to $\{T \subseteq B \mid f(a) \in T\}$.

Example 1.8. Let $Ex : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}$ be the exchange functor, i.e, $Ex(A, B) = (B, A)$ and $Ex(f, g) = (g, f)$ and $(-) \times (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ be the product functor. Then, the assignment $s : (-) \times (-) \Rightarrow [(-) \times (-)] \circ Ex$ defined by $s_{(A,B)} : A \times B \to B \times A$ as $s_{A \times B}(a, b) = (b, a)$ is a natural transformation:

$$
\begin{array}{ccc}
A \times B & \xrightarrow{s_{(A,B)}} & B \times A \\
{\scriptstyle f \times g}\downarrow & & \downarrow{\scriptstyle g \times f} \\
C \times D & \xrightarrow[s_{(C,D)}]{} & D \times C
\end{array}
\qquad
\begin{array}{ccc}
(a, b) & \xmapsto{s_{(A,B)}} & (b, a) \\
{\scriptstyle f \times g}\downarrow & & \downarrow{\scriptstyle g \times f} \\
(f(a), g(b)) & \xmapsto[s_{(C,D)}]{} & (g(b), f(a))
\end{array}
$$

Exercise 1.9. Prove that $\alpha : ((-) \times (-)) \times (-) \Rightarrow (-) \times ((-) \times (-))$ defined by $\alpha_{A,B,C} : (A \times B) \times C \to A \times (B \times C)$ such that $\alpha_{A,B,C}((a, b), c) = (a, (b, c))$ is a natural transformation.

Example 1.10. The assignment $(-)^{-1}(1) : Hom(-, 2) \Rightarrow P^{\circ}(-)$ defined by

$f \mapsto f^{-1}$ is a natural transformation:

$$\begin{array}{ccc} Hom(B,2) & \xrightarrow{(-)^{-1}(1)} & P(B) \\ {\scriptstyle (-)\circ f}\downarrow & & \downarrow{\scriptstyle f^{-1}(-)} \\ Hom(A,2) & \xrightarrow[(-)^{-1}(1)]{} & P(A) \end{array} \qquad \begin{array}{ccc} g & \xmapsto{(-)^{-1}(1)} & g^{-1}(1) \\ {\scriptstyle (-)\circ f}\downarrow & & \downarrow{\scriptstyle f^{-1}(-)} \\ gf & \xmapsto[(-)^{-1}(1)]{} & (gf)^{-1}(1) \end{array}$$

Example 1.11. The assignment $(-)(0) : Hom(1,-) \Rightarrow id_{\mathbf{Set}}$ defined by $g \mapsto g(0)$ is a natural transformation:

$$\begin{array}{ccc} Hom(1,A) & \xrightarrow{(-)(0)} & A \\ {\scriptstyle f\circ(-)}\downarrow & & \downarrow{\scriptstyle f} \\ Hom(1,B) & \xrightarrow[(-)(0)]{} & B \end{array} \qquad \begin{array}{ccc} g & \xmapsto{(-)(0)} & g(0) \\ {\scriptstyle f\circ(-)}\downarrow & & \downarrow{\scriptstyle f} \\ fg & \xmapsto[(-)(0)]{} & fg(0) \end{array}$$

Similarly, for the category of groups, we have:

$$\begin{array}{ccc} Hom(\mathbb{Z},G) & \xrightarrow{(-)(1)} & U(G) \\ {\scriptstyle f\circ(-)}\downarrow & & \downarrow{\scriptstyle f} \\ Hom(\mathbb{Z},H) & \xrightarrow[(-)(1)]{} & U(H) \end{array} \qquad \begin{array}{ccc} g & \xmapsto{(-)(1)} & g(1) \\ {\scriptstyle f\circ(-)}\downarrow & & \downarrow{\scriptstyle f} \\ fg & \xmapsto[(-)(1)]{} & fg(1) \end{array}$$

where $U : \mathbf{Grp} \to \mathbf{Set}$ is the forgetful functor, $(-)(1) : Hom(\mathbb{Z},-) \Rightarrow U$ defined as $g \mapsto g(1)$. We also have the same phenomenon in $\mathbf{Vec}_{\mathbb{R}}$, i.e.,

$$\begin{array}{ccc} Hom(\mathbb{R},V) & \xrightarrow{(-)(1)} & U(V) \\ {\scriptstyle f\circ(-)}\downarrow & & \downarrow{\scriptstyle T} \\ Hom(\mathbb{R},W) & \xrightarrow[(-)(1)]{} & U(W) \end{array} \qquad \begin{array}{ccc} g & \xmapsto{(-)(1)} & g(1) \\ {\scriptstyle T\circ(-)}\downarrow & & \downarrow{\scriptstyle T} \\ Tg & \xmapsto[(-)(1)]{} & Tg(1) \end{array}$$