Mathematical Structuralism, S09

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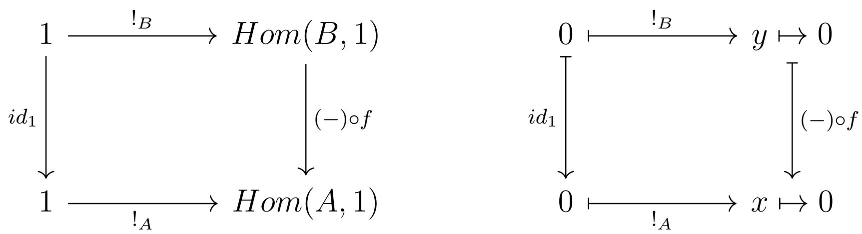
1 Category Theory (continued)

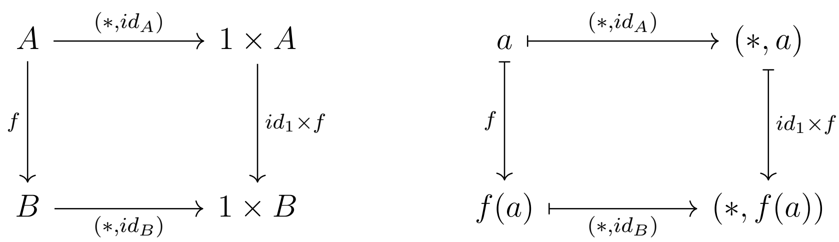
1.1 Functors and Natural Transformations

Example 1.1. The assignment $! : \Delta_1 \Rightarrow Hom(-, 1)$ defined by $!_A(0) = cons_A$ is a natural transformation, where $cons_A : A \to 1$ is the constant function, mapping everything to zero:



Exercise 1.2. Prove that $\alpha : p_1 \Rightarrow Hom(-, -)$ defined by $\alpha_{A,B} : A \to Hom(B, A)$ such that $\alpha_{A,B}(a) = cons_{A,B,a}$ is a natural transformation, where $p_1 : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$ is the projection on the second element functor and $cons_{A,B,a} : B \to A$ maps every element in B to a.

Example 1.3. The assignment $(*, id_{(-)}) : id_{\mathbf{Set}} \Rightarrow 1 \times (-)$ defined by $a \mapsto (*, a)$ is a natural transformation:



Example 1.4. Let B and C be two fixed sets. Then, the assignment $\alpha : (-)^{B+C} \Rightarrow (-)^B \times (-)^C$ defined by $\alpha_A(g) = (g|_B, g|_C)$ is a natural transformation:

$$\begin{array}{ccc} A^{B+C} & \xrightarrow{\alpha_A} & A^B \times A^C \\ {\scriptstyle f\circ(-)}\downarrow & & \downarrow{\scriptstyle f\circ(-)\times f\circ(-)} \\ A'^{B+C} & \xrightarrow[\alpha_B]{} & A'^B \times A'^C \end{array} \qquad \begin{array}{ccc} g & \xmapsto{\alpha_A} & (g|_B, g|_C) \\ {\scriptstyle f\circ(-)}\downarrow & & \downarrow{\scriptstyle f\circ(-)\times f\circ(-)} \\ fg & \xmapsto[\alpha_B]{} & (fg|_B, fg|_C) \end{array}$$

Exercise 1.5. Prove that $\alpha : (-)^{(-)+(-)} \Rightarrow (-)^{(-)} \times (-)^{(-)}$ defined by $\alpha_{A,B,C}(g) = (g|_B, g|_C)$ is a natural transformation.

Exercise 1.6. Prove that $\alpha : ((-) \times (-))^{(-)} \Rightarrow (-)^{(-)} \times (-)^{(-)}$ defined by $\alpha_{A,B,C} : (A \times B)^C \to A^C \times B^C$ such that $\alpha_{A,B,C}(g) = (p_0 \circ g, p_1 \circ g)$ is a natural transformation, where $p_0 : A \times B \to A$ and $p_1 : A \times B \to B$ are the projection functions.

Exercise 1.7. Prove that $\alpha : Hom((-), (-)^{(-)}) \Rightarrow Hom(- \times -, -)$ defined by $\alpha_{A,B,C} : Hom(A, C^B) \to Hom(A \times B, C)$ such that $\alpha_{A,B,C}(g) = \hat{g}$ is a natural transformation, where $\hat{g} : A \times B \to C$ maps (a, b) to $g(a)(b)$.

Philosophical Note 1.8. Looking inside the world of categories, there are three sorts of data: First, the categories as the nodes or the zero-dimensional data; the functors between the categories as the edges or the 1-dimensional data and finally, natural transformations as the surfaces or the 2-dimensional data. In this sense, the world of categories is at least 2-dimensional in some intuitive sense. It is possible to make all these considerations more precise to even show that this world is exactly 2-dimensional and there is no non-trivial data beyond natural transformations. Unfortunately, this task is far beyond the scope of our first chapter.

Example 1.9. (*Non-natural transformations*) Let α be an assignment of a map $\alpha_A : A \to A$ to any set A. Then, $\alpha : id_{\mathbf{Set}} \Rightarrow id_{\mathbf{Set}}$ is a natural transformation iff $\alpha_A = id_A$. It is clear that $\alpha_A = id_A$ is a natural transformation. For the converse, assume α is a natural transformation and consider the following commutative diagram:

$$\begin{array}{ccc} \{0\} & \xrightarrow{\alpha_{\{0\}}} & \{0\} \\ {\scriptstyle \hat{a}}\downarrow & & \downarrow{\scriptstyle \hat{a}} \\ A & \xrightarrow[\alpha_A]{} & A \end{array}$$

where $a \in A$ and $\hat{a}(0) = a$. It is clear that $\alpha_{\{0\}} = id_{\{0\}}$. Hence, $\alpha_A \circ \hat{a} = \hat{a}$. Applying both sides on 0, we have $\alpha_A(a) = a$. Hence, $\alpha_A = id_A$.

Exercise 1.10. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism iff there exists a functor $G : \mathbf{Set} \to \mathbf{Set}$ such that $F \circ G = G \circ F = id_{\mathbf{Set}}$ and in this situation $\mathcal{C}$ and $\mathcal{D}$ is called isomorphic. Then, prove that if $\mathcal{E}$ and $\mathcal{F}$ are two categories isomorphic to **Set** and $F_0, F_1 : \mathcal{E} \to \mathcal{F}$ be two isomorphisms, then there exists exactly one natural transformation from F to F'.

Philosophical Note 1.11. Reading the groupoid of categories as a 2-dimensional space, Exercise 1.10 implies that the connective component of the category **Set** is 1-dimensional, as the level of natural transformations collapses in this component. Moreover, it has no holes, as any space between two isomorphisms can be filled by a natural transformation.

Example 1.12. Let G and H be two groups and $F, F' : G \to H$ be two group homomorphisms. Then, a natural transformation $\alpha : F \Rightarrow F'$, by definition is an element $\alpha_* = h \in H$ such that $F(g)h = hF'(g)$, for any $g \in G$.

$$\begin{array}{ccc} F(*) & \xrightarrow{\alpha_*} & F'(*) \\ {\scriptstyle F(g)}\downarrow & & \downarrow{\scriptstyle F'(g)} \\ F(*) & \xrightarrow[\alpha_*]{} & F'(*) \end{array}$$

Exercise 1.13. Let $\mathcal{C}$ be a category. By the center of $\mathcal{C}$, denoted by $Z(\mathcal{C})$, we mean the class of all natural transformation $\alpha : id_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$. Show that for any group G considered as a category, $Z(G)$ corresponds to the set $\{g \in G \mid \forall h \in G \; gh = hg\}$. Use this characterization to show that any non-trivial abelian group G have at least two different natural transformations over its identity isomorphism. Moreover, find a group G such that between any two isomorphisms over G, there is at most one morphism.

Exercise 1.14. Prove that there is no natural transformation from $id_{(\mathbb{Z},+)}$ to $-id_{(\mathbb{Z},+)}$.

Philosophical Note 1.15. Exercise 1.13 implies that the connective component of any non-trivial abelian group is truly 2-dimensional, while for a group G that $Z(G) = \{e\}$, its component is just 1-dimensional. This may explain why the abelian groups are easier to work with, or more generally, why the groups become more complex, as soon as their centers start to shrink.

Example 1.16. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets and $F, G : (P, \leq_P) \to (Q, \leq_Q)$ be two poset morphisms. Then, a natural transformation $\alpha : F \Rightarrow G$ is neccessarily unique, as there is at most one map from $F(p)$ to $G(p)$, for any $p \in P$. This unique natural transformation exists iff $F(p) \leq_Q G(p)$, for any $p \in P$:

$$\begin{array}{ccc} F(p) & \overset{\alpha_p}{\leq} & G(p) \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ F(q) & \underset{\alpha_q}{\leq} & G(q) \end{array}$$

Exercise 1.17. Construct a poset $(P, \leq)$ and an isomorphism $F : (P, \leq) \to (P, \leq)$ such that there is no natural transformations $\alpha : id_{(P,\leq)} \Rightarrow F$ and $\beta : F \Rightarrow id_{(P,\leq)}$.

Philosophical Note 1.18. Consider the category of all locally small categories with isomorphism as the morphisms. Then, the previous considerations imply that the "*topological*" picture of this category must be like: