Mathematical Structuralism, S10

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1 Category Theory (continued)

1.1 Functors and Natural Transformations

Example 1.1. Let G be a group. Recall that a G-action is a group homomorphism from G to $Aut(X)$, where $Aut(X)$ is the group of all bijections on the set X. A morphism between two G-actions is a function $\phi : X \to Y$ such that $\phi(F(g)(x)) = F'(g)(\phi(x))$, for any $g \in G$ and $x \in X$. Then, any G-action is just a functor $G \to \mathbf{Set}$ and any morphism between two G-actions is a natural transformation:

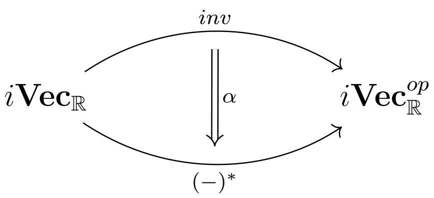
$$\begin{CD}
X = F(*) @>{\phi}>> F'(*) = Y \\
@V{F(g)}VV @VV{F'(g)}V \\
X = F(*) @>>{\phi}> F'(*) = Y
\end{CD}$$

Example 1.2. Let $(-)^* : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Vec}_{\mathbb{R}}^{op}$ be the functor mapping V to $V^* = \{T : V \to \mathbb{R} \mid T \text{ is linear}\}$ and $S : V \to W$ to $(-) \circ S : W^* \to V^*$. Then, the assignment $i : id_{\mathbf{Vec}_{\mathbb{R}}} \Rightarrow ((-)^*)^*$ defined by $i_V(v) : Hom(V, \mathbb{R}) \to \mathbb{R}$ as $i_V(v)(T) = T(v)$ is a natural transformation:

$$\begin{CD}
V @>{i_V}>> V^{**} \\
@V{T}VV @VV{T^{**}}V \\
W @>>{i_V}> W^{**}
\end{CD}
\qquad\qquad
\begin{array}{ccc}
v & \xmapsto{\;i_V\;} & [S \mapsto S(v)] \\
{\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T^{**}} \\
T(v) & \xmapsto[\;i_W\;]{} & [R \mapsto R(T(v))]
\end{array}$$

because, if we spell out the definition of $T^{**} : V^{**} \to W^{**}$, we see $T^{**}(F)(f) = F(f \circ T)$, where $F \in Hom(V, \mathbb{R}) \to \mathbb{R}$ and $f \in Hom(W, \mathbb{R})$.

Remark 1.3. It is well-known that any finite-dimensional vector space V is isomorphic to its dual V^*, using the map $\alpha_V(v) = \hat{v}$, where $\hat{v}(w) = \langle w, v\rangle$ in which $\langle -, -\rangle$ is the usual inner product. This transformation is not natural, simply because the functors $id_{\mathbf{Vec}_\mathbb{R}} : \mathbf{Vec}_\mathbb{R} \to \mathbf{Vec}_\mathbb{R}$ and $(-)^* : \mathbf{Vec}_\mathbb{R} \to \mathbf{Vec}_\mathbb{R}^{op}$ don't have the same codomain. One may find this reason quite artificial, as the map i_V seems quite natural, indeed. To address this issue, let us restrict ourselves to the subcategory of $\mathbf{Vec}_\mathbb{R}$, where all morphism are isomorphisms. Denote this subcategory by $i\mathbf{Vec}_\mathbb{R}$. Then, it is possible to make the directions right, using the functor $inv : i\mathbf{Vec}_\mathbb{R} \to i\mathbf{Vec}_\mathbb{R}^{op}$ that fixes the objects and maps any isomorphism to its inverse. Now, we have the following *possibly* natural transformation:



However, it is still not natural, as if we check the naturality condition, it requires:

$$\begin{array}{ccc} W & \xrightarrow{\alpha_W} & W^* \\ {\scriptstyle T^{-1}}\downarrow & & \downarrow{\scriptstyle (-)\circ T} \\ V & \xrightarrow[\alpha_V]{} & V^* \end{array} \qquad \begin{array}{ccc} w & \xmapsto{\alpha_W} & [u \mapsto \langle u, w\rangle] \\ {\scriptstyle T^{-1}}\downarrow & & \downarrow{\scriptstyle (-)\circ T} \\ T^{-1}(w) & \xmapsto[\alpha_V]{} & [v \mapsto \langle v, T^{-1}(w)\rangle] \end{array}$$

meaning, $\langle T(v), w\rangle = \langle v, T^{-1}(w)\rangle$, which is not the case. One can easily check that this equation holds for any $T : V \to W$ that preserves the inner product. Therefore, if we restrict the categories more to invertible linear maps that preserve inner product (*orthogonal transformations*), then our assignment α finally will be a natural transformation. Note that this restricted category actually captures the Euclidean geometry as it works with maps that respect distance and angle. Therefore, we can read the naturality of α as "angles are natural in Euclidean geometry, while they are not in linear world".

Example 1.4. (*No-deleting theorem*) There is only one natural transformation $\alpha : (-) \times (-) \to pr_1$. This natural transformation is the trivial $\alpha_{B,A} = \emptyset$. First, it is clear that this assignment is a natural transformation. Conversely, assume such an α exists. Then, we have the following commutative diagram:

$$\begin{array}{ccc} B \times A & \xrightarrow{\alpha_{B,A}} & A \\ {\scriptstyle S\times R}\big\downarrow & & \big\downarrow{\scriptstyle R} \\ B \times A & \xrightarrow[\alpha_{B,A}]{} & A \end{array}$$

for any set A and B and any relations $R \subseteq A^2$ and $S \subseteq B^2$. Set $R = \{(a, a) \mid a \in A\}$ and $S = \emptyset$. Then, $\alpha_{B,A}$ must be empty. It is useful to check why the usual projection function does not work in this case. If we spell out all the details, the reason boils down to the fact that the relations can be partial.

Example 1.5. (*No-cloning theorem*) There is only one natural transformation $\alpha : id_{\mathbf{Rel}} \to (-)^2$, where $(-)^2 : \mathbf{Rel} \to \mathbf{Rel}$ is defined by $A \mapsto A^2$ on objects and $R \mapsto R \times R$ on morphisms. This natural transformation is the trivial $\alpha_A = \emptyset$. First, it is clear that this assignment is a natural transformation. Conversely, assume that $\alpha : id_{\mathbf{Rel}} \to (-)^2$ is a natural transformation. Then, we have the following commutative diagram:

$$\begin{array}{ccc} \{0\} & \xrightarrow{\alpha_{\{0\}}} & \{0\}^2 \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle R^2} \\ A & \xrightarrow[\alpha_A]{} & A^2 \end{array}$$

for any set A and any relation $R \subseteq A \times \{0\}$. First, note that $\alpha_{\{0\}} \subseteq \{0\} \times \{0\}^2$. As $\{0\} \times \{0\}^2$ has just one element, then either $\alpha_{\{0\}} = \emptyset$ or $\alpha_{\{0\}} = \{0\} \times \{0\}^2$. The first case implies that $\alpha_A = \emptyset$, for every A. If R_A is non-empty, then there exists $(a, (b, c)) \in \alpha_A$ for some $a, b, c \in A$. Define $R = \{(0, a)\}$. Then, $(0, (b, c)) \in \alpha_a \circ R$, while $R^2 \circ \alpha_{\{0\}}$ is empty. Hence, $\alpha_A = \emptyset$. For the second case, we prove that $\alpha_{\{0\}} = \{0\} \times \{0\}^2$ is impossible. First, set $R = \{(0, a)\}$. Then, $R^2 \circ \alpha_{\{0\}} = \{(0, (a, a))\}$. Therefore, $\alpha_A \circ R = \{(0, (a, a))\}$ which means $(a, (a, a)) \in \alpha_A$. Therefore, $\{(a, (a, a)) \mid a \in A\} \subseteq \alpha_A$. It is easy to prove that α_A can not have any other element and hence $\alpha_A = \{(a, (a, a)) \mid a \in A\}$. Now, set $A = \{0, 1\}$ and $R = \{0\} \times A$. We have $\alpha_A \circ R = \{(0, (0, 0)), (0, (1, 1))\}$. But, $R^2 \circ \alpha_{\{0\}} = \{0\} \times A^2$ which is a contradiction. It is useful to check why the usual function $a \mapsto (a, a)$ does not work. If we spell out all the details, the reason boils down to the fact that the relations can be multi-valued.

Philosophical Note 1.6. The simplest category that encodes the quantum behavior is **Rel**, in which sets encode the set of states and relations encode the non-deterministic processes that change one state to another. In this sense, the previous two theorems are the baby version of the *entanglement* phenomenon in quantum theory by which we know it is impossible to clone or delete a quantum bit of information. The reason for the simplest case of **Rel** may be explained by the fact that relations can be partial or multi-valued and this makes the elements of the set somewhat entangled to each other. The more advanced version states that there is no natural transformation $\alpha_V : V \to V \bigotimes V$ or $\beta_{V,W} : V \bigotimes W \to V$ on vector spaces. For the real version, replace vector spaces by Hilbert spaces and linear maps by bounded linear maps.

Exercise 1.7. Let $List : \mathbf{Set} \to \mathbf{Set}$ be the functor mapping any set X to the set of all finite sequences of the elements of X and mapping any function $f : X \to Y$ to the function $List(f) : List(X) \to List(Y)$ defined by $List(f)(\sigma_0 \cdots \sigma_n) = f(\sigma_0) \cdots f(\sigma_n)$. Show that the assignment $i : \Delta_1 \to List$ defined by $i_X : \{0\} \to List(X)$ as $i_X(0) = \epsilon$ is a natural transformation, where ϵ is the sequence with the length zero. Moreover, show that the assignment $m : List \times List \to List$ defined by $m_X : List(X) \times List(X) \to List(X)$ as the concatenation operation is a natural transformation.

Example 1.8. Let $\mathcal{B}$ be the groupoid of finite sets and bijections. Define $Aut : \mathcal{B} \to \mathbf{Set}$ as the functor mapping any set X to the set of all bijections on X and mapping a bijection $f : X \to Y$ to the function $f \circ (-) \circ f^{-1} : Aut(X) \to Aut(Y)$. Moreover, define $Ord : \mathcal{B} \to \mathbf{Set}$ as the functor mapping any set X to the set of all finite sequences of the elements of X in which any element of X occurs exactly once. For the morphisms, map a bijection $f : X \to Y$ to the function $Ord(f) : Ord(X) \to Ord(Y)$ defined as $Ord(f)(\sigma_0 \cdots \sigma_n) = f(\sigma_0) \cdots f(\sigma_n)$. Then, there is no natural transformation $\alpha : Aut \to Ord$. Because, if there is such a transformation, then:

$$\begin{array}{ccc} Aut(X) & \xrightarrow{\alpha_X} & Ord(X) \\ {\scriptstyle f\circ(-)\circ f^{-1}}\downarrow & & \downarrow{\scriptstyle Ord(f)} \\ Aut(X) & \xrightarrow[\alpha_X]{} & Ord(X) \end{array}$$

for any set X and any bijection $f : X \to X$. Set X as a set with at least two elements and $f : X \to X$ as a non-identity bijection. Now, apply the diagram on $id_X \in Aut(X)$. We have $\alpha_X(f id_X f^{-1}) = Ord(f)(\alpha_X(id_X))$ which means $\alpha_X(id_X) = Ord(f)(\alpha_X(id_X))$. This implies that $\alpha_X(id_X)$ is a list of all the elements of X that does not change under the application of f. Hence, f must be the identity function which is a contradiction. Note that in this example, although for any finite set X, the sets $Ord(X)$ and $Aut(X)$ are isomorphic, there is no natural transformation between Aut and Ord as construction methods. Specially, it means that the isomorphisms between $Ord(X)$ and $Aut(X)$ is not natural in X.

Example 1.9. Let $\mathcal{R}_{in}$ be the category $\mathcal{R}$ of Example **??**, restricted to injective homomorphism. Let $GL_n : \mathcal{R}_{in} \to \mathbf{Grp}$ be the functor mapping any object R to the group of all invertible $n \times n$ matrices with entries in R and any morphism $f : R \to S$ to $GL_n(f) : GL_n(R) \to GL_n(S)$ defined as $GL_n(f)(A) = f[A]$, where $f[A]$ is the result of the application of f on all the entries of A. Note that $GL_n(f)(A)$ is well-defined, because, if A is invertible, then so is $f[A]$. The reason is that if $f(det(A)) = det(f[A]) = 0 = f(0)$, then $det(A) = 0$, as f is injective which implies that A is not invertible. Moreover, note that the assignment $det : GL_n \Rightarrow GL_1$ is a natural transformation. The reason is that the determinant of a matrix is a polynomial in the entries of the matrix and hence it is preserved by the morphisms of $\mathcal{R}$:

$$\begin{array}{ccc} GL_n(R) & \xrightarrow{det_R} & R \\ {\scriptstyle GL_n(f)}\downarrow & & \downarrow{\scriptstyle f} \\ GL_n(S) & \xrightarrow[det_S]{} & S \end{array}$$

Example 1.10. Let $U : \mathbf{Mon} \to \mathbf{Set}$ and $F : \mathbf{Set} \to \mathbf{Mon}$ be the forgetful and the free functors, respectively. Then, the assignments $i : id_{\mathbf{Set}} \Rightarrow UF$ mapping a set A to the function $i_A : X \to UF(X)$ defined by $i_A(x) = x$ is a natural transformation. Similarly, the assignments $p : FU \Rightarrow id_{\mathbf{Mon}}$ mapping a monoid M to the homomorphism $p_M : FU(M) \to M$ defined by $p_M(\sigma_0 \cdots \sigma_n) = \sigma_0 \times \cdots \times \sigma_n$ is a natural transformation.

Example 1.11. Let $\mathcal{C}$ be a category and $f : A \to B$ be a map. Then, the assignment $y_f : Hom(-, A) \to Hom(-, B)$ defined by $(y_f)_C = f \circ (-)$ is a natural transformation:

$$\begin{array}{ccc} Hom(D, A) & \xrightarrow{f \circ (-)} & Hom(D, B) \\ {\scriptstyle (-)\circ g}\downarrow & & \downarrow{\scriptstyle (-)\circ g} \\ Hom(C, A) & \xrightarrow[f \circ (-)]{} & Hom(C, B) \end{array}$$