Mathematical Structuralism, S11

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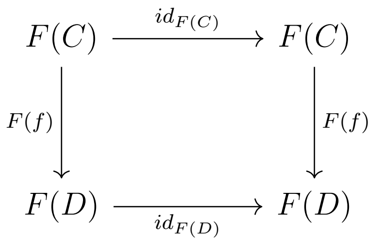
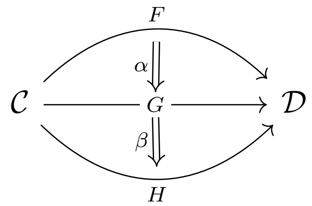
1 Category Theory (continued)

1.1 Functors and Natural Transformations

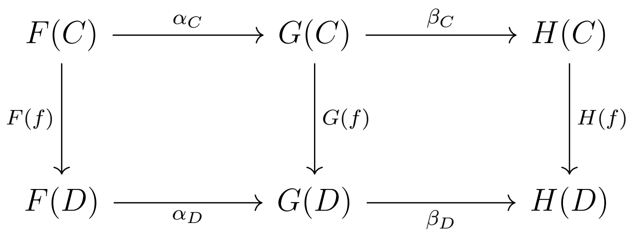
Example 1.1. Let $\mathcal{C}$ be a category, $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then, the assignment $\alpha : F \Rightarrow F$ defined by $\alpha_C = id_{F(C)}$ is a natural transformation, because:



Example 1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $F, G, H : \mathcal{C} \to \mathcal{D}$ be three functors and $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be two natural transformations, then $\beta \circ \alpha : F \Rightarrow H$, defined by $(\beta \circ \alpha)_C = \beta_C \alpha_C$ is a natural transformation:



Because in the following diagram:

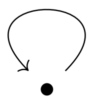


if both of the squares commute, the bigger rectangular also commutes.

Definition 1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Then, the functors from $\mathcal{C}$ to $\mathcal{D}$ as the objects together with the natural transformations as the morphisms constitutes a category. This category is denoted by $\mathcal{D}^{\mathcal{C}}$ and is called a functor category.

Remark 1.4. Note that if $\mathcal{C}$ and $\mathcal{D}$ are both small categories, then $\mathcal{D}^{\mathcal{C}}$ is also small. If $\mathcal{C}$ is small and $\mathcal{D}$ is locally small, then $\mathcal{D}^{\mathcal{C}}$ is locally small. But if $\mathcal{C}$ and $\mathcal{D}$ are both locally small, there is no reason for $\mathcal{D}^{\mathcal{C}}$ to be locally small and it is usually not the case.

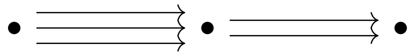
Example 1.5. The category of variable sets $\mathbf{Set}^{\rightarrow}$, the category of dynamical spaces $\mathbf{Set}^{\circlearrowright}$ and the category of G-actions are the functor categories $\mathbf{Set}^{\mathbf{2}}$, $\mathbf{Set}^{\mathbf{S}}$ and $\mathbf{Set}^{G}$, respectively, where $\mathbf{S}$ is the following category:



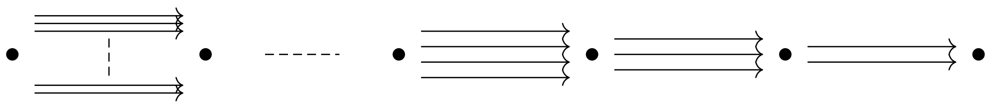
Note that $\mathbf{Set}^{\circlearrowright}$ is just $\mathbf{Set}^{(\mathbb{N},+)}$. As two other examples, note that $\mathcal{C}^{\rightarrow}$ is the functor category $\mathcal{C}^{\mathbf{2}}$ and if we consider the set $n = \{0, \cdots, n-1\}$ as a discrete category, $\mathcal{C}^{n}$ is essentially the same as the category $\mathcal{C} \times \mathcal{C} \times \cdots \mathcal{C}$, where the number of $\mathcal{C}$'s is n.

Philosophical Note 1.6. There is a philosophical shift in considering functor categories, as it treats functors or more philosophically "construction methods" as the objects of the discourse, themselves.

Example 1.7. The category of quivers is the functor category $\mathbf{Set}^{\rightrightarrows}$. In a similar way, the category of 2-quivers is $\mathbf{Set}^{\Delta_2^{nd}}$, where Δ_2^{nd} is the following category:



Similarly, we can imagine the category of n-quivers as $\mathbf{Set}^{\Delta_n^{nd}}$, where Δ_n^{nd} is the following category:



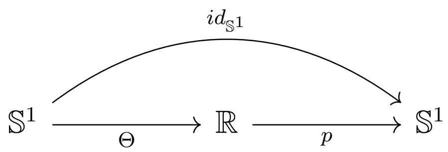
with $n+1$ objects and $i+1$ primitive morphisms between the ith and $i+1$th objects, counted from the right. What is the category of ∞-quivers?

The previous examples lead to a general notion of diagram. Intuitively, a diagram is a set of objects together with a set of morphisms between them in a given category $\mathcal{C}$. More formally, though:

Definition 1.8. Let $\mathcal{J}$ and $\mathcal{C}$ be two categories. Then, a functor $D : \mathcal{J} \to \mathcal{C}$ is called a diagram in $\mathcal{C}$ with shape $\mathcal{J}$ or a $\mathcal{J}$-diagram in $\mathcal{C}$. Therefore, the functor category $\mathcal{C}^{\mathcal{J}}$ is called the category of diagrams in $\mathcal{C}$ with shape $\mathcal{J}$.

Example 1.9. (*Algebra*) What is an algebraic construction, only using the algebraic concepts? It is reasonable to assume that an algebraic construction, whatever it is, must be available for all the algebras in consideration and it must respect the algebraic maps. In this sense, if we choose the category $\mathcal{R}$ as the world of algebra, then the functor category $\mathbf{Set}^{\mathcal{R}}$ can be considered as the world of all algebraic constructions. In this category we have all V_I's (the roots of the polynomial equations in I). In this sense, V_I may be considered as the extension of the set of integers by the roots of the given polynomials in I. This mindset is the extension of the usual approach of extending the number systems by adding the solutions of the equations and hence we can think of $\mathbf{Set}^{\mathcal{R}}$ as the *ultimate completion* of the algebra $\mathbb{Z}$. Interestingly, there are more algebraic notions than what we get by adding the roots of polynomials. For instance, the functor $\mathbb{P} : \mathcal{R} \to \mathbf{Set}$ defined by $\mathbb{P}(R) = \{L \subseteq R^2 \mid L \text{ is a line}\}$ and $\mathbb{P}(f)(L) = f(L)$ is a functor, where by a line $L \subseteq R^2$ we mean the set of the roots of a linear equation $ax + by = 0$, for $a, b \in R$ and by $f(L)$ we mean the line define by the equation $f(a)z + f(b)w = 0$. We have to check that $\mathbb{P}$ is well-defined, as the equation of a line is not uniquely determined by the line itself. However, it is easy to see that the equations $ax + by = 0$ and $cx + dy = 0$ define the same line iff $(a, b) = \lambda(c, d)$, for some $\lambda \in R$. This proves that $\mathbb{P}$ is well-defined. The functor $\mathbb{P}$ corresponds to the projective space $\mathbb{P}(\mathbb{Z})$, which is again a completion of $\mathbb{Z}$ by adding the *points at infinity* it lacks.

Example 1.10. (*Topology*) Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle with its usual topology. First, let us show that it is impossible to find a continuous way to compute the angle between the point $a \in \mathbb{S}^1$ as a vector and the positive part of the x-axis. Formally, it means that there is no continuous function $\Theta : \mathbb{S}^1 \to \mathbb{R}$ such that:

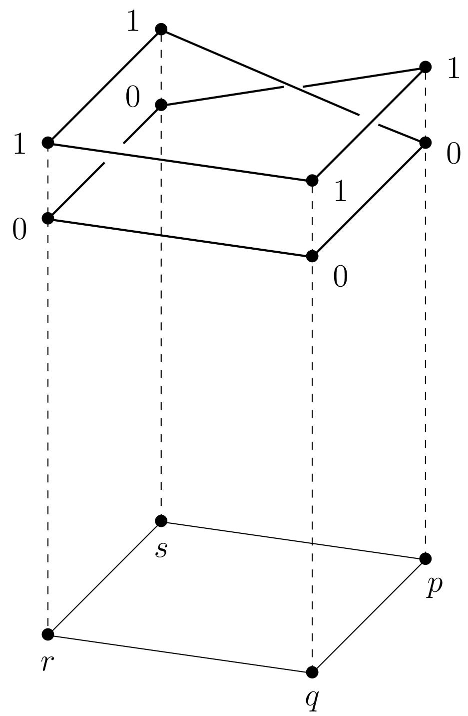


where $p : \mathbb{R} \to \mathbb{S}^1$ is the continuous function $p(\theta) = (cos\theta, sin\theta)$, mapping an angle to its corresponding point. The reason is again the argument we used for Brouwer's fixed point theorem. Since $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ is a functor, if we pick an arbitrary $a \in \mathbb{S}^1$, we have:

$$\overbrace{\pi_1(\mathbb{S}^1, a) \xrightarrow[\pi_1(\Theta)]{} \pi_1(\mathbb{R}, \Theta(a)) \xrightarrow[\pi_1(p)]{} \pi_1(\mathbb{S}^1, a)}^{id_{\pi_1(S_1,a)}}$$

which is impossible, as $\pi_1(\mathbb{S}^1, a)$ is isomorphic to $\mathbb{Z}$, while $\pi_1(\mathbb{R}, \Theta(a))$ is a singleton group.
Although, we just provided a proof, it feels paradoxical that a such continuous map does not exist. The reason is that if we restrict ourselves to a local neighborhood U of a point on $\mathbb{S}^1$, there is clearly a continuous angle map on U and since the continuity is a local notion, we expect to have a continuous map in the end. What is wrong? The problem is that the angle is continuous as long as we consider it as a multi-valued function. Let us explain why by Starting from $(1, 0)$ and moving along the circle counterclockwise. If we set the angle zero at the beginning, then it continuously grows from zero to 2π. Reaching the starting point again, if we want to remain continuous, the angle should be 2π which is impossible, as it has been set to zero before. The space is too *twisted* to have a continuous single-valued angle. To capture the true nature of the angle function, we must accept that it really is a multi-valued function, defined as an assignment mapping the point $a \in \mathbb{S}^1$ to the set $\{\theta \in \mathbb{R} \mid p(\theta) = a\}$. Now, based on the argument we had, we expect Θ to be continuous. But, what does it mean to have a continuous set-valued function? Here is an idea. For the usual functions, we can observe that they are continuous iff their restrictions to the subspaces of the space can be *glued* together. We can use the same idea here to say that a set-valued function is continuous if its restrictions to the subspaces of the space can be glued together in a reasonable generalized sense. For now, our machinery is not mature enough to talk about this gluing notion. However, we are ready to appreciate the fact that this generalized notion of continuity, whatever it is, needs the set-valued angle function to be defined on all subspaces of the space $\mathbb{S}^1$ and not just on the points. In our case, the natural definition is $\Theta : P(\mathbb{S}^1)^{op} \to \mathbf{Set}$ defined by $\Theta(X) = \{f : X \to \mathbb{R} \mid pf = id\}$. This Θ is a functor, if we map the inclusion function in $P(\mathbb{S}^1)$ to the restriction function in $\mathbf{Set}$. Hence, it is reasonable to think of the category $\mathbf{Set}^{P(\mathbb{S}^1)^{op}}$ as the world of all multi-valued functions inside of which the world of continuous multi-valued functions exists.

Example 1.11. (*Logic*) Let $\Phi = \{p \leftrightarrow q, q \leftrightarrow r, r \leftrightarrow s, s \leftrightarrow \neg p\}$ be a set of formulas. Clearly, Φ is inconsistent and has no models. Similar to the previous example, here again, the situation is a bit paradoxical. First, the set is *locally consistent* in the following sense: for any proper subset X of the set $\{p, q, r, s\}$, the part of Φ that constructed only from the atoms in X, denoted by Φ_X, is consistent. Secondly, if a valuation does not satisfy the whole set, it must behave inconsistent at some atom, where it must be forced to both zero and one. Hence, the inconsistency must be a local notion, while the set is locally consistent. To see how it is similar to the previous example, let us try to find a model for Φ. If we set the value $a \in \{0, 1\}$ for the atom p, then to satisfy Φ, the atoms q, r and s must have the same value a. Then, reaching p again, we can see that it must have the value $1 - a$ to remain consistent while the value has been set to a. The set Φ is too *twisted* to have a single-valued model:



Again, one can say that Φ has a model, but this model is multi-valued. To capture that multi-valued nature, we must use functors again. Define the generalized model, not only on points, but also on all subsets. We have $V : P(\{p, q, r, s\})^{op} \to \mathbf{Set}$ defined by $V(X) = \{v : X \to \{0, 1\} \mid v \text{ satisfies } \Phi_X\}$. This is again a functor. Hence, it is reasonable to think of $\mathbf{Set}^{P(\{p,q,r,s\})^{op}}$ as the world of all generalized models for the formulas constructing from these atoms.

Philosophical Note 1.12. One might object that as joyful as the previous approach to the inconsistencies is, it is simply empty, as it is actually impossible to have a real inconsistency in the real world. First, in our weak defence, it is worth mentioning that in practice, it usually happens that we have some local mistakes in some extremely huge database and we obviously do not want to get rid of the whole dataset because of a local mistake probably even in some other far way parts of our database. This twisted valuations is a natural way to handle such locally consistent yet globally inconsistent database. In our strong defence, though, these inconsistencies really happen in the nature and even better, the previous example is the logical version of a real situation. More precisely, assume that p, q, r and s are four quantum bits in a way that $\{p, q\}$, $\{q, r\}$, $\{r, s\}$ and $\{s, p\}$ are co-measurable, while it is impossible to measure all the quantum bits altogether. One may object that this does not solve the problem, as we can measure any two co-measurable bits to see that the value of p must be both zero and one. There are two ways to explain that. First, that the quantum bits and hence the physical quantities do not have any objective value, independent from the context and the measurements we do to observe them. Therefore, in different measurements, the quantum bit value may become zero or one. More provocatively, we can solve the inconsistency by saying that *the objective real world does not exist.* The second approach, though is that to accept the new generalized valuations as some sort of new reality. In this apparently better scenario, we might say that our usual models for reality are insufficient and we must simply model the world by these multi-valued quantities. The price to pay is now the non-locality of the reality, as these new models are global and twisted.