Mathematical Structuralism, S13

Amir Tabatabai

March 11, 2021

1 Category Theory (continued)

1.1 Functors and Natural Transformations

Example 1.1. (*Set Theory*) One of the prominent foundational paradigms in mathematics is Brouwerian intuitionism. Among many other things, the paradigm believes that mathematics is just a mental story told by a creative subject to herself and like any other story, this story is also changing through time by adding new constructions and proving new properties. In this sense, the truth in mathematics is temporal and dynamic and hence can be characterized by our variable sets in $\mathbf{Set}^{\mathcal{C}}$, where $\mathcal{C}$ is a suitable category that encodes the growth of time. There are many valid formalizations of this notion of time. For instance, the simplest formalization that comes to mind is the set of all natural numbers and its usual order encoding the instances and the arrow of time. However, in this example we focus on Brouwer's own understanding of time:

> *This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics.* [**?**]

To formalize this notion of time, we use $[n] = \{0, 1, \ldots, n-1\}$, for $n \geq 0$, as the objects to represent the nth moment of time and for any $n \leq m$, we define the morphisms from $[n]$ to $[m]$ as a function $f : [m] \to [n]$ where $f(i) = i$, for any $i < n$. The equation $f(i) = j$ represents the creation process of the moments by encoding the fact that the moment i has been created from the moment j. Therefore, the condition $f(i) = i$ just says that when we are at the nth moment, the moment $i < n$ is fixed throughout the creation process and only the moments greater than or equal to n are newly created. As it is expected, the category $\mathbf{Set}^{\mathcal{C}}$ leads to an interesting intuitionistic dynamic version of sets. What is surprising, though, is the fact that some of these variable growing sets are in some sense *completed* and the category of these completed sets satisfies all classical axioms of set theory except the axiom of choice. Hence, it can serve as a model to prove the unprovability of the axiom of choice from **ZF**.

1.2 Baby Erlangen extended

How to interpret the objects of the category $\mathbf{Set}^{\mathcal{C}^{op}}$? We saw that a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ is a $\mathcal{C}^{op}$-variable set or a realization of the category $\mathcal{C}^{op}$ using the usual concrete sets. Now, we add another interpretation to the league. Interpret $\mathcal{C}$ as the category of some sort of interesting yet simple objects and then read a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ as an ideal object identifiable by the set of the "maps" going from the simple object A in $\mathcal{C}$ to the ideal object F. Note that the category $\mathcal{C}$ is considered to be too small with too simple objects to have the ideal object F and hence the set $F(A)$ of "maps" from A to F is not a priori meaningful. However, whatever these sets are, they must behave in a functorial way and hence it is reasonable to think of any functor as the way we describe the ways the category of lenses in $\mathcal{C}$ looks inside of F. To have an intuitive example, we can think of $\mathcal{C}$ as the category with a single object $\mathbb{R}$ and continuous functions over it. Then, we can interpret $\mathcal{C}$ as the category consisting of one flat one-dimensional line and the new ideal object as the circle $\mathbb{S}^1$ that is not flat and hence lives outside of $\mathcal{C}$. However, as the circle is locally homeomorphic with $\mathbb{R}$, we can identify it by the continuous functions from $\mathbb{R}$ to it. In other words, if I know all possible maps from $\mathbb{R}$ to $\mathbb{S}^1$, then I know the space $\mathbb{S}^1$.

Now, as we interpret a functor F as a generalized ideal object $\mathcal{C}$-object, it is reasonable to replace even the simple objects of $\mathcal{C}$ by the functors that capture their behavior. In other words, if functors are ideal objects, the usual objects must be among them. as well. This is what the Yoneda functor does:

Definition 1.2. (*Yoneda functor*) Let $\mathcal{C}$ be a locally small category. Define the Yoneda functor $y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$ by $y_A = Hom(-, A)$ on objects and on the morphism $f : A \to B$ by $y_f : Hom(-, A) \to Hom(-, B)$, where $(y_f)_C : Hom(C, A) \to Hom(C, B)$ defined by $(y_f)_C(g) = fg$. A functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ is called representable, if there exists an object A in $\mathcal{C}$ such that $F \cong y_A$.

Theorem 1.3. *The Yoneda functor is actually a functor.*

Proof. First, recall that the map y_f is a natural transformation, for any map $f : A \to B$, as we have:

$$\begin{array}{ccc} Hom(D, A) & \xrightarrow{(y_f)_D = f\circ(-)} & Hom(D, B) \\ \Big\downarrow {\scriptstyle Hom(g,A)=(-)\circ g} & & \Big\downarrow {\scriptstyle Hom(g,B)=(-)\circ g} \\ Hom(C, A) & \xrightarrow[(y_f)_C = f\circ(-)]{} & Hom(C, B) \end{array}$$

Now, to prove that y is a functor, we need to show that $y_{id} = id$ and $y_{fg} = y_f y_g$. Both claim are clear by the definition of the Yoneda functor on morphisms. □

Remark 1.4. Changing $\mathcal{C}$ to $\mathcal{C}^{op}$, it is equally natural to have the dual functor $y^{(-)} : \mathcal{C}^{op} \to \mathbf{Set}^{\mathcal{C}}$, defined by $y^A = Hom(A, -)$ and $(y^f)_C(g) = gf$. It is also customary to call a functor $F : \mathcal{C} \to \mathbf{Set}$ representable if $F \cong y^A$, for some object A in $\mathcal{C}$.

Example 1.5. The functors $id_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ and $P^\circ : \mathbf{Set}^{op} \to \mathbf{Set}$ are representable, because $id_{\mathbf{Set}} \cong Hom(1, -)$ and $P^\circ \cong Hom(-, \{0, 1\})$.

Example 1.6. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is representable, because $U \cong Hom(1, -)$, where $1 = \{0\}$ is the trivial topological space. Also, the functor $\mathcal{O} : \mathbf{Top}^{op} \to \mathbf{Set}$ defined on objects by $\mathcal{O}(X)$ as the set of the open subsets of X and on morphisms by $\mathcal{O}(f) = f^{-1}$, is representable, because $\mathcal{O} \cong Hom(-, S)$, where S is the *Serpienski space* that is the space $\{0, 1\}$ with the opens $\{\emptyset, \{1\}, \{0, 1\}\}$.

Example 1.7. The forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ is representable, because $U \cong Hom(\mathbb{N}, -)$. Similarly, the forgetful functors $V : \mathbf{Grp} \to \mathbf{Set}$ and $W : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$ are representable, because $V \cong Hom(\mathbb{Z}, -)$ and $W \cong Hom(\mathbb{R}, -)$.

Example 1.8. Let A and B be two fixed sets. The functor $Hom(A, -) \times Hom(B, -) : \mathbf{Set} \to \mathbf{Set}$ is representable, because $Hom(A, -) \times Hom(B, -) \cong Hom(A + B, -)$.

Example 1.9. Let G and H be two fixed groups. The functor $Hom(-, G) \times Hom(-, H) : \mathbf{Grp}^{op} \to \mathbf{Set}$ is representable, because $Hom(-, G) \times Hom(-, H) \cong Hom(-, G \times H)$.

Example 1.10. The functor $T_n : \mathbf{Grp} \to \mathbf{Set}$ mapping any group G to $\{x \in G \mid x^n = e\}$ and any homomorphism to its appropriate restriction is representable, because $T_n \cong Hom(\mathbb{Z}_n, -)$.

Example 1.11. Let U and V be two fixed vector spaces. Then, the functor $Bilin_{U,V} : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$ defined by $Bilin_{U,V}(W) = \{T : U \times V \to W \mid T \text{ is bilinear}\}$ and composition, is representable, because $Bilin_{U,V} \cong Hom(U \otimes V, -)$.

Philosophical Note 1.12. The last example has some special illuminating role. Pedagogically, tensor product with its relatively complex construction is hard to grasp for the newcomers. To solve this issue, sometimes it is helpful to replace its detailed uninformative construction with the functor it represents, namely $Bilin_{U,V}$. This is a point in usual Borbaki-style algebra that we need to make a shift from what the objects actually *are* to what they practically *do*. We can safely pretend that the only thing that we know about the tensor product $U \otimes V$ is that it is a vector space with the property that the linear maps going out of it *naturally* correspond to the bilinear maps going out from $U \times V$. This technique of replacing the huge construction of an object with what it does is the simplest example of what we can call the structuralism in action.

Highlighting the importance of representables, it is now natural to ask if there is a criterion to check whether a given functor is representable or not. We approach this problem slowly. First, three examples:

Example 1.13. The functor $\Delta_2 : \mathbf{Set} \to \mathbf{Set}$ mapping all objects to $2 = \{0, 1\}$ and all morphisms to identity is *not* representable. Because, if $\Delta_2 \cong Hom(A, -)$, then since $Hom(A, B \times C) \cong Hom(A, B) \times Hom(A, C)$, we must have $\Delta_2(B \times C) \cong \Delta_2(B) \times \Delta_2(C)$ which means $\{0, 1\} \times \{0, 1\} \cong \{0, 1\}$.

Example 1.14. Let G and H be two groups such that there are at least two homomorphisms from G to H. Then, the functor $Hom(- \times G, H) : \mathbf{Grp}^{op} \to \mathbf{Set}$ is *not* representable. Because, if $Hom(- \times G, H) \cong Hom(-, K)$, then since $Hom(\{e\}, K)$ has just one element, the set $Hom(\{e\} \times G, H) \cong Hom(G, H)$ must have one element which is impossible by assumption.

Example 1.15. The functor $Sub : \mathbf{Grp}^{op} \to \mathbf{Set}$ mapping a group to the set of its subgroups and a morphism to the inverse image is *not* representable. Because, if $Sub(-) \cong Hom(-, K)$, then since $Hom(G \oplus H, K) \cong Hom(G, K) \times Hom(H, K)$, we have to have $Sub(G \oplus H) \cong Sub(G) \times Sub(H)$. The last statement is false, because $Sub(\mathbb{Z}_2) \times Sub(\mathbb{Z}_2)$ has exactly four elements while $Sub(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ has at least five elements including all the elements of $Sub(\mathbb{Z}_2) \times Sub(\mathbb{Z}_2)$ plus the subgroup $\{(0, 0), (1, 1)\}$.

In the general situation, there is a criteria to check the representability of a functor, imitating what we saw in the previous two examples. The main idea is that the Hom functor preserves some sort of construction (in our examples product, the "smallest possible" object, and the direct sum, respectively) and if a functor is representable, it must preserve these structures, as well. We will introduce these structures to see when this preservation can be even sufficient for representability. For now, let us focus our main story of interpreting functors as ideal objects.