Mathematical Structuralism, S14

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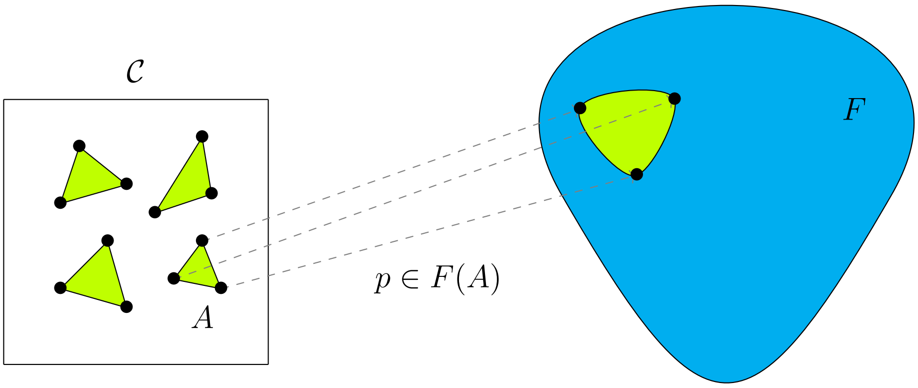
1 Category Theory (continued)

1.1 Baby Erlangen extended



So far, we have provided a way to interpret the objects of $\mathcal{C}$ as the ideal objects embodied as functors. Now, we have two things to check. First, we have to make sure that the behavior of these new copies in their new world is the same as their behavior in their original world. This means that we have to show that the Yoneda functor is a full and faithful functor, also called an embedding. Secondly, if $F : \mathcal{C}^{op} \to \mathbf{Set}$ is an ideal object and if $F(A)$ encodes the set of all "maps" from A to F, then moving to the new world of $\mathbf{Set}^{\mathcal{C}^{op}}$ where there is a copy of A, namely y_A, and also there is a well-defined notion of map from this copy to F, stored in $Hom(y_A, F)$, we expect to have a canonical isomorphism between $Hom(y_A, F)$ and $F(A)$. This expectation is fortunately a theorem and it is called the Yoneda lemma. We first prove this lemma and then we will use it to prove the fullness and faithfulness of $y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$.

Theorem 1.1. *(The Yoneda lemma) The functors* $Hom(y_{(-)}, -) : \mathcal{C}^{op} \times \mathbf{Set}^{\mathcal{C}^{op}} \to \mathbf{Set}$ *and* $(-)(-) : \mathcal{C}^{op} \times \mathbf{Set}^{\mathcal{C}^{op}} \to \mathbf{Set}$ *are naturally isomorphic via the maps* $\alpha_{A,F} : Hom(y_A, F) \to F(A)$ *and* $\bar{\alpha}_{A,F} : F(A) \to Hom(y_A, F)$ *defined by* $\alpha_{A,F}(\beta) = \beta_A(id_A)$ *and* $[\bar{\alpha}_{A,F}(p)]_C(f) = F(f)(p)$*. Specially,* $Hom(y_A, F) \cong F(A)$*, natural in* A *and* F*.*

Proof. We have to show that α and $\bar{\alpha}$ are natural transformations and for each A and F the maps $\alpha_{(A,F)}$ and $\bar{\alpha}_{A,F}$ are the inverse of each other in **Set**. For the first, note that $\beta = \bar{\alpha}_{A,F}(p)$ is a natural transformation because

$$\begin{array}{ccc} Hom(D,A) & \xrightarrow{\beta_D} & F(D) \\ {\scriptstyle Hom(g,A)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ Hom(C,A) & \xrightarrow[\beta_C]{} & F(C) \end{array}$$

But $F(g)\beta_D(f) = F(g)F(f)(p) = F(gf)(p)$. For naturality, we just check the naturality for α. The naturality of $\bar{\alpha}$ will be the result of the fact that it is the pointwise inverse of α. For α, we have to show that for any map $f : B \to A$ and any $\gamma : F \Rightarrow G$:

$$\begin{array}{ccc} Hom(y_A,F) & \xrightarrow{\alpha_{(A,F)}} & F(A) \\ {\scriptstyle Hom(y_f,\gamma)}\downarrow & & \downarrow{\scriptstyle G(f)\gamma_A = \gamma_B F(f)} \\ Hom(y_B,G) & \xrightarrow[\alpha_{(B,G)}]{} & G(B) \end{array}$$

It is not hard to prove the commutativity of the diagram and we will leave this tiresome task to the reader. For the second part, note that any $\beta \in Hom(y_a, F)$ is uniquely determined by $\beta_A(id_A)$. The reason is the following naturality diagram, for a map $f : C \to A$:

$$\begin{array}{ccc} Hom(A,A) & \xrightarrow{\beta_A} & F(A) \\ {\scriptstyle Hom(f,A)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ Hom(C,A) & \xrightarrow[\beta_C]{} & F(C) \end{array}$$

which implies that for any $f : C \to A$, we have $\beta_C(f) = F(f)(\beta_A(id_A))$. This shows that $\alpha_{\bar{A},F}\alpha_{A,F}(\beta) = \bar{\alpha}_{A,F}(\beta_A(id_A)) = \beta$ as both $\bar{\alpha}(\beta_A(id_A))$ and β on C and f are $\beta_C(f) = F(f)(\beta_A(id_A))$. For the converse, we simply have $\alpha_{A,F}\bar{\alpha}_{A,F}(p) = F(id_A)(p) = p$. □

Corollary 1.2. *(The Yoneda embedding) The functor* $y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$ *is full and faithful.*

Proof. By the Yoneda lemma, the map $\bar{\alpha}_{A,y_B} : y_B(A) = Hom(A, B) \to Hom(y_A, y_B)$ is a natural isomorphism. Computing $\bar{\alpha}_{A,y_B}$, we see that

$$[\bar{\alpha}_{A,y_B}(f)]_C(g) = y_B(f)(g) = fg = [y_f]_C(g),$$

for any C and $g : C \to A$. Hence, $\bar{\alpha}_{A,y_B}(f) = y_f$. Therefore, the map $y_{(-)} : Hom(A, B) \to Hom(y_A, y_B)$ is a bijection which means that $y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$ is a full and faithful functor. □

Philosophical Note 1.3. Note that this embedding is a representation theorem stating that any abstract category can be seen as a category of variable sets. This is useful, as the category $\mathbf{Set}^{\mathcal{C}^{op}}$ is a category of sets with set-like behavior. Hence, whenever we want to investigate something about $\mathcal{C}$, we can embed it into $\mathbf{Set}^{\mathcal{C}^{op}}$ to have enough set-theoretic machinery. Then, if we finally reach a representable functor, we can come back to the original category we started with.

Corollary 1.4. *(Uniqueness of the representing object)* $y_A \cong y_B$ *iff* $A \cong B$. *The same holds for* $y^{(-)}$.

Proof. Since the Yoneda functor is full and faithful and for any such functor F, we have $F(A) \cong F(B)$ iff $A \cong B$, the claim follows. □

Philosophical Note 1.5. From the philosophical point of view, the uniqueness of the representing object means that the relative data of an object is enough to identify it. Therefore, whenever it is convenient, we forget the object and work with its functor.

Example 1.6. Using the relative behavior of tensor products, we prove that it is commutative, i.e., $U \otimes V \cong V \otimes U$ and $\mathbb{R} \otimes V \cong V$. We have

$$Hom(U \otimes V, W) \cong Bilin_{U,V}(W) \cong Bilin_{V,U}(W) \cong Hom(V \otimes U, W)$$

natural in W. Hence, $y^{U \otimes V} \cong y^{V \otimes U}$ which implies $U \otimes V \cong V \otimes U$. With the same line of reasoning, we have

$$Hom(\mathbb{R} \otimes V, W) \cong Bilin_{\mathbb{R},V}(W) \cong Hom(V, W)$$

natural in W. Hence, $y^{\mathbb{R} \otimes V} \cong y^V$ which implies $\mathbb{R} \otimes V \cong V$.

We will see more applications later, but first, we want to use our new machinery to define some new categorical objects by identifying the relative behavior that we expect them to have. Then, the uniqueness of the representing object ensures that the defined object is unique up to isomorphism. To that purpose, it is convenient to provide an equivalent characterization of the representable functors by one of the core notions of category theory, namely the *universality*.

Theorem 1.7. *(Universal elements) A functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ is representable iff there exists an object A in $\mathcal{C}$ and an element $a \in F(A)$ such that for any object B and any element $b \in F(B)$, there exists a unique $f : B \to A$ such that $F(f)(a) = b$. The object A and the element $a \in F(A)$ are called the universal object and the universal element, respectively.*

Proof. By Yoneda lemma, an element $a \in F(A)$ corresponds to the natural transformation $\beta : y_A \Rightarrow F$, defined by $\beta_C(f) = F(f)(a)$. Note that β is a natural isomorphism iff β_C is an isomorphism for all C. The latter is exactly what the universality condition says. $\square$

Philosophical Note 1.8. If we read F as a structured set, then $a \in F(A)$ may be interpreted as the *generic point* of *the generic structure* that can act as all structures and all elements generically.

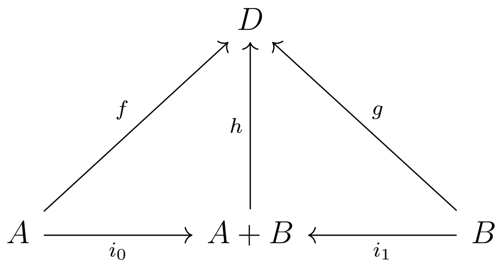
Remark 1.9. Note that the universal pair (A, a) if exists is unique up to isomorphism, i.e., if both (A, a) and (B, b) are universal for F, then there exists an isomorphisms $f : B \to A$ such that $F(f)(a) = b$. Why?

Example 1.10. For the functor $P^{\circ} : \mathbf{Set}^{op} \to \mathbf{Set}$, the universal element is $\{1\} \in P(\{0, 1\})$. The universality condition states that any set $U \in P(X)$ is obtainable by applying $P^{\circ}(f) = f^{-1}$ on $\{1\}$, for a unique $f : X \to \{0, 1\}$. This unique function is the characteristic function of U in X.

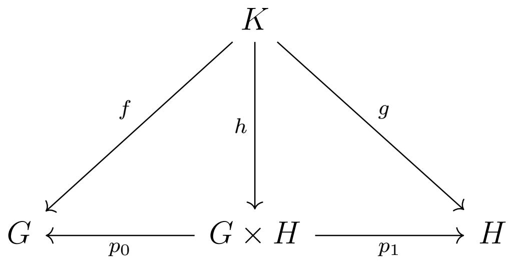
Example 1.11. For the forgetful functors $U : \mathbf{Mon} \to \mathbf{Set}$, $V : \mathbf{Grp} \to \mathbf{Set}$ and $W : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$, the universal elements are $1 \in U(\mathbb{N}) = \mathbb{N}$, $1 \in V(\mathbb{Z}) = \mathbb{Z}$ and $1 \in W(\mathbb{R}) = \mathbb{R}$. We just explain the case of monoids. The reason is that for any element $m \in U(M)$, there exists a unique monoid homomorphism $f : \mathbb{N} \to M$ such that $U(f)(1) = f(1) = m$.

Example 1.12. For the functor $Hom(A, -) \times Hom(B, -) : \mathbf{Set} \to \mathbf{Set}$, the universal element is $(i_0, i_1) \in Hom(A, A + B) \times Hom(B, A + B)$, where $i_0 : A \to A + B$ is defined by $i_0(a) = (0, a)$ and $i_1 : B \to A + B$ is defined by $i_1(b) = (1, b)$. The reason is that for any set C and any element $(f, g) \in$

$Hom(A, C) \times Hom(B, C)$, there exists a unique map $h : A + B \to C$ such that $[Hom(A, h) \times Hom(B, h)](i_0, i_1) = (hi_0, hi_1) = (f, g)$, i.e.,



Example 1.13. For the functor $Hom(-, G) \times Hom(-, H) : \mathbf{Grp}^{op} \to \mathbf{Set}$, the universal element is $(p_0, p_1) \in Hom(G \times H, G) \times Hom(G \times H, H)$, where p_0 and p_1 are the projections. The reason is that for any group K and any element $(f, g) \in Hom(K, G) \times Hom(K, H)$, there exists a unique map $h : K \to G \times H$ such that $[Hom(h, G) \times Hom(h, H)](p_0, p_1) = (p_0 h, p_1 h) = (f, g)$, i.e.,



Example 1.14. For the functor $Bilin_{U,V} : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$, the universal element is $i \in Bilin(U \otimes V)$, where $i : U \times V \to U \otimes V$ is defined by the bilinear function $i(u, v) = u \otimes v$. The universality condition says that for any element $f : Bilin_{U,V}(W) = \{f : U \times V \to W \mid f \text{ is bilinear}\}$, there exists a unique linear map $g : U \otimes V \to W$ such that $Bilin(g)(i) = gi = f$, i.e.,