Mathematical Structuralism, S15

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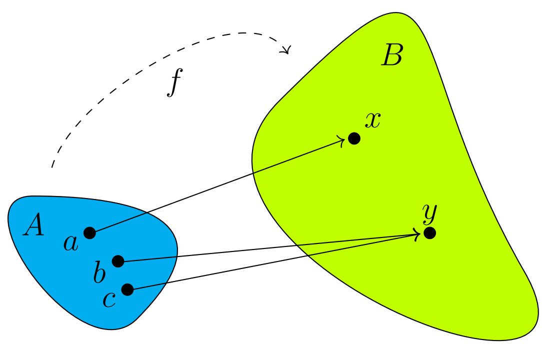
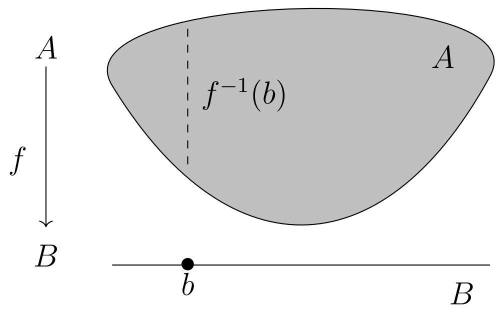
1 Category Theory (continued)

1.1 Baby Erlangen extended

Philosophical Note 1.1. There are two ways to interpret a function $f : A \to B$ in **Set**. First, as an A-indexed element of B or simply an A-element of B, reading a parameter $a \in A$ to output $f(a) \in B$:



Here we are labelling the elements of B by A. In the second interpretation, we read a map $f : A \to B$ as a B-indexed family of subsets of A, a B-subset of A or just a fibration over B, mapping $b \in B$ to the set (fiber) $f^{-1}(b) \subseteq A$:



Here we are stacking the elements of A by B. Thanks to Yoneda embedding, it is reasonable to lift these interpretations to any arbitrary category, by interpreting objects as variable sets and morphisms as variable functions. This way, we can interpret a map $f : A \to B$ as some sort of A-element of B, reading a parameter $a : X \to A$ to output $fa : X \to B$ or as some sort of B-part of A or a fibration over B, reading a parameter $b : X \to B$ to output the fiber $\{a : X \to A \mid fa = b\}$.

These two interpretations are useful in different settings. Usually, in a category, we have some small simple known objects and to know any arbitrary object A, we investigate the maps to/from A from/to these simple objects. For instance, in geometry, we investigate a geometrical object by the maps *from* the Euclidean cubes or the higher dimensional balls *into* it, while in algebra, we study an algebraic object by more relations we can put on its elements transforming the algebra *to* simpler algebras of the same kind. These two dual approaches is what distinguish geometrical from algebraic way of thinking. In some cases, it is possible to see both of the approaches at the same time. For instance, living in **Set**, as $\{0\}$ and $\{0, 1\}$ are simple, we can study X *geometrically* by all the maps going from $\{0\}$ to X, i.e., its elements, while investigating X by the maps from X to $\{0, 1\}$ is the *algebraic* study of X via the boolean algebra of its subsets. A similar situation happens in algebraic geometry, logic and functional analysis. In the first, we can study a polynomial equation either by working in the polynomial algebra modulo the equation or by the zeros the equation has in some choice of simple rings such as algebraically closed fields. In logic we have syntax versus semantics and in functional analysis we can study a topological space either by looking inside the topology or by working with its function algebra as the world of measurable quantities over the space.

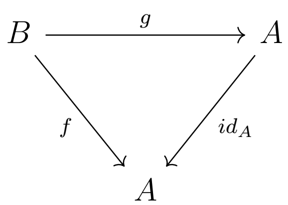
Finally, note that using these two interpretations, if we interpret A as our interesting object in a category $\mathcal{C}$, the slice category $\mathcal{C}/A$ is the category of all fibration over A, while the coslice category $A/\mathcal{C}$ is the category of A-enhanced objects having a distorted copy of A inside.

Now, we are ready to define some categorical constructions by representability or equivalently by universality.

Definition 1.2. An object A is called terminal if it represents the functor $\Delta_1 : \mathcal{C}^{op} \to \mathbf{Set}$, i.e., $Hom(B, A) \cong \{0\}$, natural in B. Equivalently, A is terminal if for any B, there exists a unique map form B to A. Since this object is unique up to isomorphism, we denote it by 1.

Example 1.3. In categories **Set**, **Grp**, **Ab**, $\mathbf{Vect}_{\mathbb{R}}$ and **Cat**, the terminal object exits and is $\{0\}$, interpreted respectively. In a poset $(P, \leq)$, the terminal object is by definition an element $a \in P$ such that for any $b \in P$, we have $b \leq a$. Hence, the terminal object is the greatest element of the poset. Any non-trivial monoid as a category does not have a terminal object, because if the only object of a monoid is terminal, then there must be exactly one morphism over that object.

Example 1.4. In the category $\mathcal{C}/A$, the terminal object is $id_A : A \to A$, as for any object $g : B \to A$, there is exactly one morphisms $g : B \to A$ such that $id_A g = f$ and that morphism is f itself.

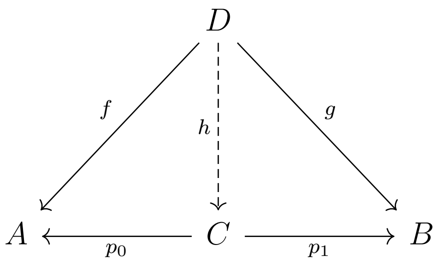


In $\mathbf{Set}/A$ the terminal object $id_A : A \to A$ corresponds to the fibration $a \mapsto \{a\}$.

Example 1.5. In the category $\mathbf{Set}^{\mathcal{C}^{op}}$, the terminal object is $\Delta_1 : \mathcal{C}^{op} \to \mathbf{Set}$, as for any functor $F : \mathcal{C}^{op} \to \mathbf{Set}$, there is exactly one natural transformation $\alpha : F \Rightarrow \Delta_1$, where $\alpha_C : F(C) \to \{0\}$ maps everything to 0.

Definition 1.6. Let A and B be two objects. An object C together with a natural isomorphism $\alpha : Hom(-, C) \cong Hom(-, A) \times Hom(-, B)$ is called a product of A and B. Equivalently, C together with two morphisms $p_0 : C \to A$ and $p_1 : C \to B$ is called a product if for any object D and any morphisms

$f : D \to A$ and $g : D \to B$, there exists a unique map $h : D \to C$ such that:



The product of A and B is denoted by $A \times B$. It is possible to extend products from the binary case to any arbitrary family. More precisely, if I is a set and $\{A_i\}_{i \in I}$ is a family of objects in $\mathcal{C}$, by their product we mean an object C together with a natural isomorphism $\alpha : Hom(-, C) \cong \Pi_{i \in I} Hom(-, A_i)$. Equivalently, it is an object C with maps $p_i : C \to A_i$ such that for any other family of maps $f_i : D \to A_i$, there exists a unique map $h : D \to C$ such that $p_i h = f_i$, for any $i \in I$. The product of $\{A_i\}_{i \in I}$ is denoted by $\Pi_{i \in I} A_i$.

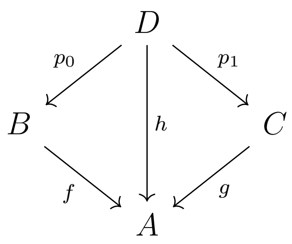
Example 1.7. In categories **Set**, **Top**, **Grp**, **Ab**, $\mathbf{Vect}_{\mathbb{R}}$ and **Cat**, the product is the usual product. In a poset $(P, \leq)$, the product of a family $\{a_i\}_{i \in I}$ is by definition the greatest lower bound of $\{a_i\}_{i \in I}$ i.e., an element c such that $c \leq a_i$ for all $i \in I$ and for any $d \in P$ if $d \leq a_i$ for all $i \in I$ then $d \leq c$. For the prototype posets, namely posets of subsets of X with inclusion, if they are closed under arbitrary intersection, the intersection of a family of subsets will be the product of the subsets. Products in posets are usually called meets and denoted by $\bigwedge$ or for finite families with $\wedge$. For the unique object $*$ in a non-trivial finite monoid as a category, even the binary product $* \times *$ does not exists, because if it does, it must be $*$ and we must have:

$$M = Hom(*, *) \cong Hom(*, * \times *) \cong Hom(*, *) \times Hom(*, *) = M \times M$$

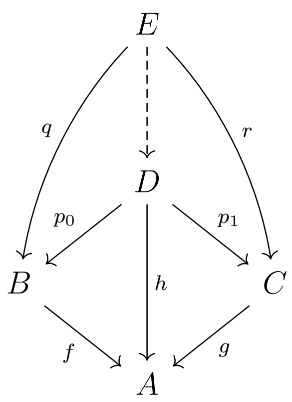
which is impossible.

Philosophical Note 1.8. When one sees the product topology for the first time, one may wonder why such a topology and its bias towards using only finite proper opens in the basis elements $\Pi_{i \in I} U_i$ is natural. Here is the answer. The product together with this topology is *the* product. For us, behaving as a product has a clear structural meaning and the object that represents this behavior may incarnate in many different forms in the different contexts. In **Top** this topology is what we have to use to have the product. Its construction, though, is secondary to what it must perform.

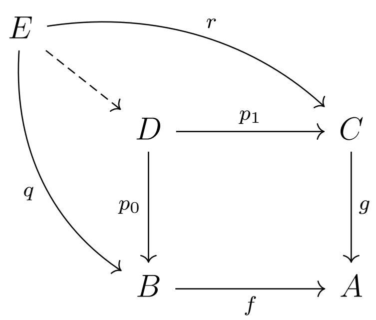
Example 1.9. (*Pullback*) What is a binary product of two objects $f : B \to A$ and $g : C \to A$ in $\mathcal{C}/A$? It is an object $h : D \to A$ and two morphism $p_0 : D \to B$ and $p_1 : D \to C$ such that:



and for any other object $e : E \to A$ and any morphisms from q from e to f and r from e to g, there exists a unique map from E to D such that:



Usually people write this data as:



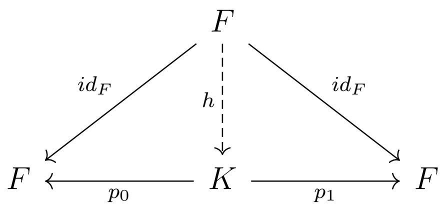
and call the square a pullback diagram, p_0 a pullback of g along f and p_1 a pullback of f along g. The pullback is also called the fiber product as it is actually the product in the category of fibrations over A. Sometimes, the object D itself is loosely called the pulllback and it is denoted by $B \times_A C$.

Example 1.10. All pullbacks exist in the category **Set**. More precisely, for the two functions $f : B \to A$ and $g : C \to A$, the pullback is $B \times_A C =$

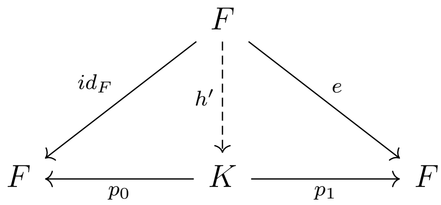
$\{(b, c) \in B \times C \mid f(b) = g(c)\}$ with the projection maps. Reading the data as fibrations, the fiber corresponding to $B \times_A C$ over $a \in A$ is nothing but $f^{-1}(a) \times g^{-1}(a)$ that is the pointwise product of fibers.

Example 1.11. In the category $\mathbf{Set}^{\mathcal{C}^{op}}$, the product of $E : \mathcal{C}^{op} \to \mathbf{Set}$ and $F : \mathcal{C}^{op} \to \mathbf{Set}$ is defined pointwise, i.e., $(E \times F)(A) = E(A) \times F(A)$ and $(E \times F)(f) = E(f) \times F(f) : E(B) \times F(B) \to E(A) \times F(A)$, for any $f : A \to B$ in $\mathcal{C}$. The projections $p_0 : E \times F \Rightarrow E$ and $p_1 : E \times F \Rightarrow F$ are also defined pointwise, i.e., $(p_0)_C : E(C) \times F(C) \to E(C)$ by projection on the first element and similarly for p_1.

Example 1.12. (*Non-existence of terminal objects and binary products*) For an easier example, consider the poset $(\mathbb{N}, \leq)$. This poset has no greatest element and hence no terminal object. For product, take the poset $(P, \subseteq)$ of all infinite subsets of $\mathbb{N}$. Then, the product (meet) of the set E of the even numbers and O of the odd numbers does not exists, as there is no infinite set below both of them. For a more interesting example, take the category of fields. This category has no terminal object, because if F is terminal, for any other field E, there must be a map from E to F. However, any map between two fields is one-to-one and hence F must have the maximum cardinality between all fields which is impossible. The binary product also does not exist. For instance, if the field $F = \mathbb{Q} \times \mathbb{Z}_p$ exists, then it has two maps one into $\mathbb{Q}$ and one into $\mathbb{Z}_p$. Since $p \cdot 1 = 0$ in $\mathbb{Z}_p$ and the maps are one-to-one, we must have $p \cdot 1 = 0$ in F and hence in $\mathbb{Q}$ which is impossible. Restricting fields to a fixed characteristic p can not solve the problem. It is enough to pick a field F with a non-identity endomorphism $e : F \to F$. (For $p = 0$, pick $F = \mathbb{C}$ and $e(z) = \bar{z}$ and for a prime p, pick F as a filed with p^2 elements and $e(x) = x^p$. In the latter case, e is not identity as the equation $x^p = x$ has at most p roots while the field has p^2 elements). Then, we claim that $F \times F$ does not exist. If it does, call it K. Then, by the universal property of the product, there is $h : F \to K$ such that:



Since $p_0 h = p_1 h = id_F$, both p_0 and p_1 are surjective. Since p_0 and p_1 are also one-to-one, they are bijections and hence h is a bijection. Since $p_0 h = p_1 h$, we have $p_0 = p_1$. Now, by the universal property of the product again, there must be $h' : F \to K$ such that:



But as $p_0 = p_1$ and $e \neq id_F$, this is impossible.