Mathematical Structuralism, S16

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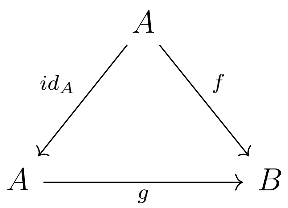
1 Category Theory (continued)

1.1 Baby Erlangen extended

Definition 1.1. An object A is called initial if it corepresents the functor $\Delta_1 : \mathcal{C} \to \mathbf{Set}$, i.e., $Hom(A, B) \cong \{0\}$, natural in B. Equivalently, A is initial if for any object B, there exists a unique map from A to B. The initial object is denoted by 0.

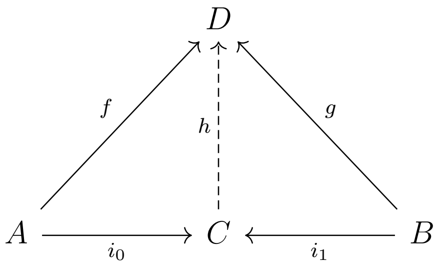
Example 1.2. In the category $\mathbf{Set}$ the initial object is the empty set. In $\mathbf{Grp}$ and $\mathbf{Vect}_{\mathbb{R}}$ it is $\{0\}$. In $\mathbf{Cat}$ it is the empty category. In a poset $(P, \leq)$, the initial object is by definition the least element. Any non-trivial monoid as a category does not have an initial object, because if the only object of a monoid is initial, then there must be exactly one morphism over that object.

Example 1.3. In the category $A/\mathcal{C}$, the initial object is $id_A : A \to A$, as for any object $f : A \to B$, there is exactly one morphisms $g : A \to B$ such that $g \circ id_A = f$. The morphism is f itself:



Example 1.4. In the category $\mathbf{Set}^{\mathcal{C}^{op}}$, the initial object is $\Delta_\emptyset : \mathcal{C}^{op} \to \mathbf{Set}$ as for any functor $F : \mathcal{C}^{op} \to \mathbf{Set}$, there is exactly one natural transformation $\alpha : \Delta_\emptyset \Rightarrow F$, that is defined by $\alpha_C : \emptyset \to F(C)$, where α_C is the empty function.

Definition 1.5. Let A and B be two objects. An object C together with a natural isomorphism $\alpha : Hom(C, -) \cong Hom(A, -) \times Hom(B, -)$ is called a coproduct of A and B. Equivalently, C together with two morphisms $i_0 : A \to C$ and $i_1 : B \to C$ is called a coproduct if for any object D and any morphisms $f : A \to D$ and $g : B \to D$, there exists a unique map $h : C \to D$ such that:



The coproduct is denoted by $A + B$. It is possible to extend coproducts from the binary case to any arbitrary family. More precisely, if I is a set and $\{A_j\}_{j \in J}$ is a family of objects in $\mathcal{C}$, by their coproduct, we mean an object C together with a natural isomorphism $\alpha : Hom(C, -) \cong \Pi_{j \in J} Hom(A_j, -)$. Equivalently, it is an object C with maps $i_j : A_j \to C$ such that for any other family of maps $f_j : A_j \to D$, there exists a unique map $h : C \to D$ such that $hi_j = f_j$, for any $j \in J$. The coproduct of $\{A_j\}_{j \in J}$ is denoted by $\Sigma_{j \in J} A_j$.

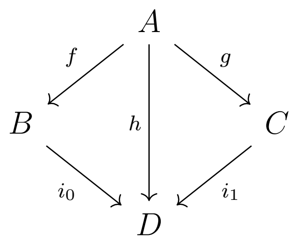
Example 1.6. In the category **Set**, the coproduct is the disjoint union with its injection functions. In **Ab** and $\mathbf{Vect}_{\mathbb{R}}$, coproduct equals to the product. In **Cat**, the coproduct is the coproduct we saw before. In a poset $(P, \leq)$, the coproduct of a family $\{a_i\}_{i \in I}$ is by definition the least upper bound of $\{a_i\}_{i \in I}$ i.e., an element c such that $a_i \leq c$, for all $i \in I$ and for any $d \in P$ if $a_i \leq d$, for all $i \in I$ then $c \leq d$. For the prototype posets, namely posets of subsets of X with inclusion, if they are closed under arbitrary union, the union of a family of subsets will be the coproduct of the subsets. Coproducts in posets are usually called joins and denoted by $\bigvee$ or for finite families with $\vee$. For the unique object $*$ in a non-trivial finite monoid as a category, the coproduct $* + *$ does not exists, because if it does, it must be $*$ and we must have:

$$M = Hom(*, *) \cong Hom(* + *, *) \cong Hom(*, *) \times Hom(*, *) = M \times M$$

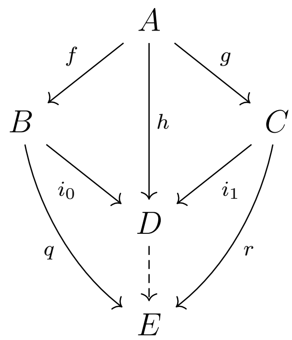
which is impossible.

Philosophical Note 1.7. When one sees the finite product in **Ab** for the first time, it may be confusing why one notion has two names, diresct sum and direct product. Later, seeing the general case, one can see the difference in general that collapses in the finite case. However, one may still wonder why we need the finiteness condition in the definition of the direct sums? Similar to what we saw for product topology, we have the same thing here. The direct sum is *the* coproduct in **Ab**. For us, behaving as a coproduct has a clear structural meaning and the object that represents this behavior may incarnate in many different forms in the different contexts. In **Ab** this group is what we have to use to have the coproduct. Its construction, though, is secondary to what it must perform.

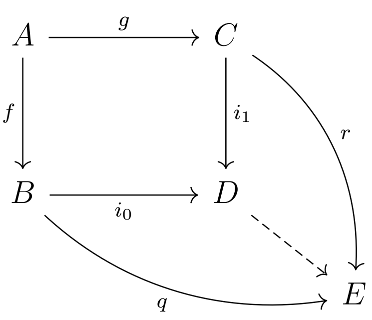
Example 1.8. (*Pushout*) What is a coproduct of two objects $f : A \to B$ and $g : A \to C$ in $A/\mathcal{C}$? It is an object $h : A \to D$ and two morphism $i_0 : B \to D$ and $i_1 : C \to D$ such that:



and for any other object $e : A \to E$ and any morphisms q from f to e and r from g to e, there exists a unique map from D to E such that:



Usually people write this data as:



and call the square a pushout square, i_1 a pushout of f along g and i_0 a pushout of g along f. The pushout is also called the cofiber coproduct as it is dual to fiber product. Sometimes, the object D itself is loosely called the pushout and it is denoted by $B +_A C$.

Example 1.9. All pushouts exist in the category **Set**. More precisely, for the two functions $f : A \to B$ and $g : A \to C$, the pushout is $B +_A C = B + C/\sim =$, where $\sim$ is the lest equivalence relation generated by $\{f(a) = g(a) \mid a \in A\}$ with the injection maps. Reading the data as A-enhanced sets, the pushout is nothing but the disjoint union of B and C in which the two copies of A are glued together. The same is also true for the category **Top** where $B + C/\sim$ is equipped with the quotient topology, i.e., the topology where U is open in $B + C/\sim$ if either $i_0^{-1}(U)$ is open in B and $i_1^{-1}(U)$ is open in C. As a concrete example, when $A = \{0\}$, the pushout is the notion of coproduct in the category of pointed spaces. For instance, $\mathbb{S}^1$ is the pushout of $f : \{0\} \to [0, 1]$ and $g : \{0\} \to [0, 1]$, where $f(0) = 0$ and $g(0) = 1$:

$$\begin{array}{ccc} \{0\} & \xrightarrow{0 \mapsto 0} & [0,1] \\ {\scriptstyle 0\mapsto 1}\downarrow & & \downarrow \\ [0,1] & \longrightarrow & \mathbb{S}^1 \end{array}$$

In **Ab**, the pushout is $B \oplus C/N$, where N is the subgroup generated by $f(a) - g(a)$'s for any $a \in A$. In **CRing**, it is $B \otimes_A C$, considering B and C as A-algebras via the maps $f : A \to B$ and $g : A \to C$.

Example 1.10. One can thinks of pushouts as scalar extensions (cobase change) in the algebraic world as the dual of the geometric base change operation. For instance, if we have an algebras structure over a field K such as $M_n(K)$, then changing the field of scalaras from K to a greater field $L \supseteq K$ is the pushout

$$\begin{array}{ccc} K & \xhookrightarrow{\quad i \quad} & L \\ {\scriptstyle a\mapsto aI_n}\downarrow & & \downarrow{\scriptstyle a\mapsto aI_n} \\ M_n(K) & \longrightarrow & M_n(K) \otimes_K L = M_n(L) \end{array}$$

Philosophical Note 1.11. For the newcomers in topology, the quotient topology is something complex and mysterious. The structural way of thinking makes it simpler by proposing that it is *the* gluing in the category of

Top. The quotient topology is just secondary to the pushout role it plays. The same holds for tensor product of A-algebras. They are just the gluing of rings as A-enhanced objects.

Philosophical Note 1.12. Structural way of thinking is useful as it shows that gluing of pointed spaces and tensor product of A-algebras for the fixed A are the same thing. Moreover, we can see that this construction is dual to the fiber product of topological spaces. Does it mean that something geometric lives in **CRing**, dully, where tensor product plays the role of fiber product?

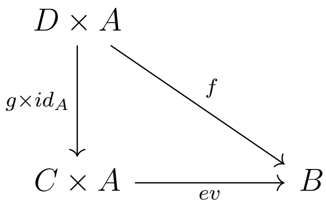
Example 1.13. In the category $\mathbf{Set}^{\mathcal{C}^{op}}$, the coproduct of $E : \mathcal{C}^{op} \to \mathbf{Set}$ and $F : \mathcal{C}^{op} \to \mathbf{Set}$ is defined pointwise, i.e., $(E + F)(A) = E(A) + F(A)$ and $(E + F)(f) = E(f) + F(f) : E(B) + F(B) \to E(A) + F(A)$, for any $f : A \to B$ in $\mathcal{C}$. The injections $i_0 : E \Rightarrow E + F$ and $i_1 : F \Rightarrow E + F$ are also defined pointwise, i.e., $(i_0)_C : E(C) \to E(C) + F(C)$ by usual set injection and similarly for i_1.

Remark 1.14. (*Duality*) Note that a terminal object in $\mathcal{C}$ is an initial object in $\mathcal{C}^{op}$ and the same also holds for the pair product/coproduct and pullback/pushout. In this sense, these pairs of notions are dual to each other or in its slogan form they are the same thing, *reversing the arrows.*

Example 1.15. (*Non-existence of initial objects and binary coproducts*) For an easier example, consider the poset $(\mathbb{Z}, \leq)$. This poset has no least element and hence no initial object. For coproduct, take the poset $(P, \subseteq)$ of all subsets of $\mathbb{N}$ whose complement is infinite. Then, the coproduct (join) of the set E of the even numbers and O of the odd numbers does not exists, as the only subset above both of them is $\mathbb{N}$ whose complement is finite. For a more interesting example, take the category of fields. This category has no initial object, because, for any other field E, there must be a map from F to E. As any map between two fields is one-to-one, the characteristics of E and F equals which excludes all E's with different characteristics. The binary product also does not exist for the same reason. Restricting fields to a fixed characteristic p can not solve the problem. The reason is similar to what we had for products before.

Definition 1.16. Let $\mathcal{C}$ be a category with products and A and B be two objects. An object C together with a natural isomorphism $\alpha : Hom(-, C) \cong Hom(- \times A, B)$ is called an exponentiation of B to A. Equivalently, an exponentiation of B to A is an object C together with a morphism $ev : C \times A \to B$ such that for any $f : D \times A \to B$, there exists a unique

$g : D \to C$ such that:



The exponentiation is denoted by B^A.

Example 1.17. In the category **Set**, the exponential is $B^A = \{f : A \to B\}$ with the morphism $ev : B^A \times A \to B$ by $ev(f, a) = f(a)$. In **Cat**, the exponential category is defined by $\mathcal{D}^{\mathcal{C}}$ as the functor category and $ev : \mathcal{D}^{\mathcal{C}} \times \mathcal{C} \to \mathcal{C}$ by $ev(F, A) = F(A)$ and $ev(\alpha, f) = \alpha_B F(f) = G(f)\alpha_A$, for any $f : A \to B$ and $\alpha : F \Rightarrow G$. The last equality is because of the naturality of α. In a poset $(P, \leq)$, the exponentiation is by definition the least element c such that $c \wedge a \leq b$ i.e., an element c such that $c \wedge a \leq b$ and for any $d \in P$ if $d \wedge a \leq b$ then $d \leq c$. For the prototype posets, namely posets of subsets of X with inclusion, if they are closed under arbitrary union and finite intersections, the exponentiation of two subsets U and V are $V^U = \bigcup\{W \in P \mid W \cap U \subseteq V\}$. Exponential objects in posets are called Heyting implications and denoted by $\to$.