Mathematical Structuralism, S17

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1 Category Theory (continued)

1.1 Baby Erlangen extended

We saw how to define categorical constructions by representability. Here, we show how these constructions are functorial.

Theorem 1.1. *Let $F : \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$ be a functor such that for any object D in $\mathcal{D}$, the functor $F(-, D)$ is representable. Then, there exists a unique (up to natural isomorphism) functor $G : \mathcal{D} \to \mathcal{C}$ such that $Hom(C, G(D)) \cong F(C, D)$, natural in C and D.*

Proof. Since for any D, the functor $F(-, D) : \mathcal{C}^{op} \to \mathbf{Set}$ is representable, there is an object $G(D)$ in $\mathcal{C}$ such that $Hom(C, G(D)) \cong_{\alpha_{C,D}} F(C, D)$, natural in C. For maps, if $f : D \to E$ is a map in $\mathcal{D}$, we define $G(f)$ as the unique morphism whose Yoneda is $y_{G(f)} = \alpha^{-1}_{C,E} F(id_C, f)\alpha_{C,D}$:

$$\begin{array}{ccc} y_{G(D)} & \xrightarrow{\alpha_{C,D}} & F(C,D) \\ {\scriptstyle y_{G(f)}}\downarrow & & \downarrow{\scriptstyle F(id_C,f)} \\ y_{G(E)} & \xrightarrow[\alpha_{C,E}]{} & F(C,E) \end{array}$$

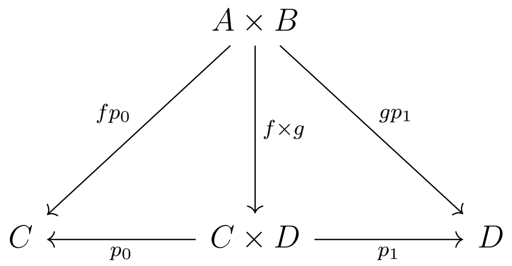
It is easy to see that G is a functor and $\alpha_{C,D}$ is also natural in D. For uniqueness, assume there are G and H have the property. Then,

$$Hom(C, G(D)) \cong F(C, D) \cong Hom(C, H(D))$$

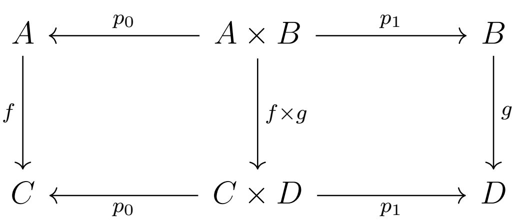
Hence, $y_{G(D)} \cong y_{H(D)}$, natural in D. By Yoneda embedding, we have $G(D) \cong H(D)$, natural in D. $\square$

Remark 1.2. Dually, if $F : \mathcal{C} \times \mathcal{D} \to \mathbf{Set}$ is a functor such that for any object D in $\mathcal{D}$, the functor $F(-, D)$ is corepresentable, there exists a unique (up to natural isomorphism) functor $G : \mathcal{D} \to \mathcal{C}^{op}$ such that $Hom(G(D), C) \cong F(C, D)$, natural in C and D.

As an application, we can see that products, coproducts and exponentials define functors. For products, it is enough to set $F : \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{C} \to \mathbf{Set}$ as $F(X, A, B) = Hom(X, A) \times Hom(X, B)$ to reach $G(A, B) = A \times B$ as the product functor. The case for coproduct is similar. For the exponential functor, set $F : \mathcal{C}^{op} \times \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ as $F(X, A, B) = Hom(X \times A, B)$ to reach $G(A, B) = B^A$ as the exponential functor.
It is always possible to provide the functor by the universal behavior that is usually tiresome. Let's do it once for product as it has some pedagogical value. Assume $f : A \to C$ and $g : B \to D$ are two morphisms and we want to define $f \times g : A \times B \to C \times D$. By the universal property of $C \times D$, it is enough to provide two maps from $A \times B \to C$ and $A \times B \to D$ and the we will have our map automatically. For these two maps, pick:

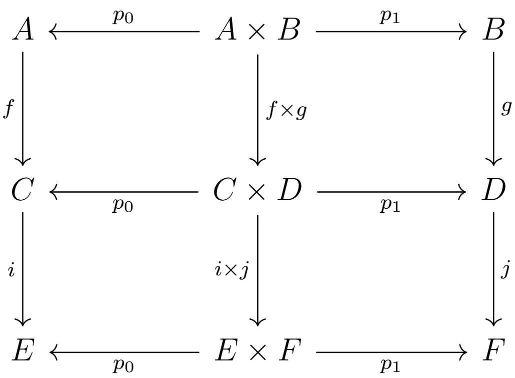


We will rewrite the previous diagram as

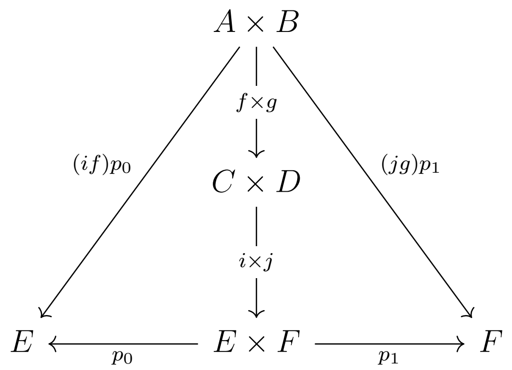


to have a more suggestive shape in our later computation. Now, we have to show that product is a functor. For that matter, assume $i : C \to E$ and

$j : D \to F$ and we have to show that $(fi) \times (gj) = (f \times g) \circ (i \times j)$. We have



Since all the small squares commute, the outer two vertical rectangular also commutes, meaning



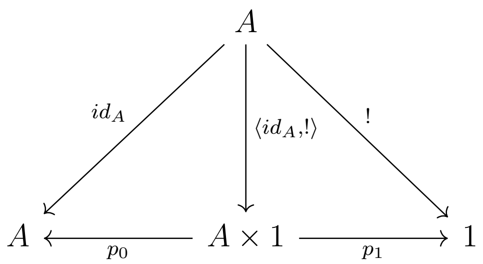
But by definition, there is only one vertical map that makes the diagram commutative, i.e., $(f \times g) \circ (i \times j)$. Hence, $(f \times g) \circ (i \times j) = (fi) \times (gj)$. The proof for $id \times id = id$ is similar.

Example 1.3. (*Yoneda lemma as a computational tool*) In any category with binary product and terminal object, we have $A \times 1 \cong A$, natural in A. As we saw before, we have to show that these two objects have the same behavior. We have

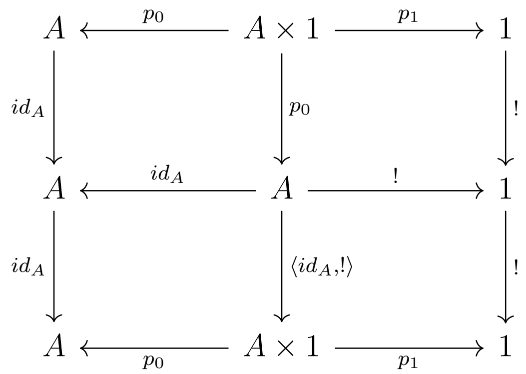
$$Hom(X, A \times 1) \cong Hom(X, A) \times Hom(X, 1) \cong Hom(X, A)$$

Hence, $y_{A\times 1} \cong y_A$ which by Yoneda lemma implies $A \times 1 \cong A$. Similarly, it is possible to prove that product is symmetric, i.e., $A \times B \cong B \times A$ and it is associative, i.e., $A \times (B \times C) \cong (A \times B) \times C$.
Again, it is possible to do the same thing by the universal property. To prove that $A \times 1 \cong A$, we must provide two maps, one from A to $A \times 1$ and one

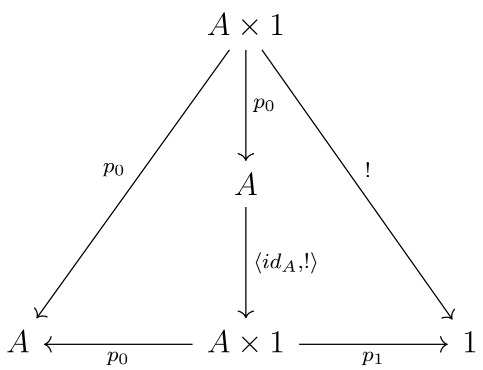
from $A \times 1$ to A such that they become each other's inverses. For these two maps, pick $f = p_0 : A \times 1 \to A$ and $g = \langle id_A, ! \rangle : A \to A \times 1$. The latter is the unique map that makes the following commutative:



It is clear that $fg = p_0 \langle id_A, ! \rangle = id_A$. For the converse, consider the following diagram



It is easy to see that all small squares are commutative and hence the outer two vertical rectangular must be commutative, meaning



But the only vertical map that makes the diagram commutative is $id_{A \times 1}$. Hence, $\langle id_A, ! \rangle p_0 = id_{A \times 1}$.

Example 1.4. (*Yoneda lemma as a computational tool*) In any category with coproduct, product and exponentiation, we have $A \times (B + C) \cong A \times B + A \times C$, natural in A, B and C. To show that these two objects have the same behavior, note that

$$Hom(A\times(B+C), D) \cong Hom((B+C), D^A) \cong Hom(B, D^A)\times Hom(C, D^A) \cong$$

$$Hom(A \times B, D) \times Hom(A \times C, D) \cong Hom(A \times B + A \times C, D)$$

Hence, $y^{A\times(B+C)} \cong y^{A\times B+A\times C}$ which by Yoneda lemma implies $A\times(B+C) \cong A \times B + A \times C$.

Example 1.5. Let $(Sub(\mathbb{R}^2), \subseteq)$ be the poset of all linear subspaces of $\mathbb{R}^2$. In this poset, all joins and meets exist. Meets are just intersections and joins are the linear subspaces generated by the unions. However, we do not have the equality $M \times (N + K) = M \times K + N \times K$ and hence the category does not have all exponentials. To show the failure of the equality, set M, N and K as three distinct lines going through the origin in $\mathbb{R}^2$. It is clear that $N + K = \mathbb{R}^2$ and hence $M \times (N + K) = M \cap (N + K) = M$, while $M \times N = M \times K = \{0\}$ and $\{0\} + \{0\} = \{0\} \neq M$.

Example 1.6. (*Non-existence of the exponential objects*) Let $\mathcal{C}$ be a non-preorder category with the initial and terminal objects where $0 \cong 1$. Then, $\mathcal{C}$ does not have all exponentials, because if it does, then we must have

$$Hom(A, B) \cong Hom(1 \times A, B) \cong Hom(1, B^A) \cong Hom(0, B^A)$$

But the last set has exactly one element. Hence $Hom(A, B)$ must have exactly one element, for any choice of A and B, which is a contradiction. As a consequence, the categories **Grp**, **Ab** and $\mathbf{Vec}_{\mathbb{R}}$ don't have all exponential objects.

Exercise 1.7. It seems that in **Ab**, the object H^G consisting of all homomorphisms from G to H with the pointwise addition is the exponential object of H by G. Find what is missing here.

Philosophical Note 1.8. (*Convenient category of spaces*) The category **Top** does not have all the exponentials and this fact makes the category somewhat cumbersome to work with. One way to overcome this issue is moving to a convenient category of topological spaces that includes a copy of all the tame interesting topological spaces like CW-complexes while having good properties including the closure under products and exponentiation. Steenrod proposed a list of such good properties for such a category. However,

> *It is also known that these propositions do not hold in the category of all Hausdorff spaces. In fact arguments have been given that which imply that there is no convenient category in our sense.*

However, Steenrod himself introduced such a category. He explains the apparent mismatch by:

> *The arguments are based on a blind adherence to the customary definitions of the standard operations. These definitions are suitable for the category of Hausdorff spaces, but they need not be for a subcategory. The categorical viewpoint enables us to defrost these definitions and bend them a bit.*

In fact, the customary definition that needs to change is the construction of the product. In Steenrod's category, a subcategory of the category of all Hausdorff spaces, all the products exist but its topology is far from the usual product topology and the adherence to this usual topology is what made the others blind to find the right category. With a bit of provocation, let's conclude that history also suggests the priority of the *relative* behavior of the entities to their *absolute* constructions.

Example 1.9. (*Yoneda lemma as a tool to define functors: subobject classifier*) Consider the functor $Sub : (\mathbf{Set}^{\mathcal{C}^{op}})^{op} \to \mathbf{Set}$ mapping a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ to the set of all sub-functors of F and a natural transformation to the pre-image function. This functor is representable by a functor $\Omega : \mathcal{C}^{op} \to \mathbf{Set}$, i.e., $Sub(F) \cong Hom(F, \Omega)$. Let's guess this functor. Using the Yoneda lemma, we know that $\Omega(A)$ must be equivalent to $Hom(y_A, \Omega)$. However, we expect $Hom(y_A, \Omega)$ to be equivalent to $Sub(y_A)$. Therefore, we can define $\Omega(A)$ as $Sub(y_A)$ and check if it really works, i.e., if $Sub(F) \cong Hom(F, \Omega)$, natural in F. We will not present the details here, but it fortunately holds.

Philosophical Note 1.10. Note that Ω plays the role of $\{0, 1\}$ in $\mathbf{Set}$. Therefore, it is reasonable to say that the object Ω is the variable set of the truth values of the new world $\mathbf{Set}^{\mathcal{C}^{op}}$. More precisely, let F be a variable set. Then, any map from F to Ω is a characteristic map of a variable subsets of F assigning truth values to the "elements" of F, according to the way that the subfunctor sits inside F. Such an Ω with this behavior is called a subobject classifier.

Example 1.11. (*Yoneda lemma as a tool to define functors: exponential object*) The category $\mathbf{Set}^{\mathcal{C}^{op}}$ has all exponential objects. To prove that, we use the Yoneda lemma again. Let $E, F : \mathcal{C}^{op} \to \mathbf{Set}$ be two functors. We need to define F^E such that $Hom(E \times X, F) \cong Hom(X, F^E)$. Again set $X = y_A$. Then, we have to have $Hom(E \times y_A, F) \cong Hom(y_A, F^E)$. But $Hom(y_A, F^E)$ must be equivalent to $F^E(A)$, by Yoneda lemma. Therefore, it is enough to define F^E by $F^E(A) = Hom(E \times y_A, F)$. The only thing to check is that if

F^E satisfies the more general $Hom(E \times X, F) \cong Hom(X, F^E)$. Again, we will not present the details here, but it fortunately holds.