Mathematical Structuralism, S18

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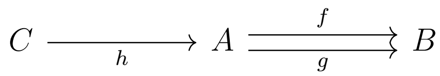
May 6, 2021

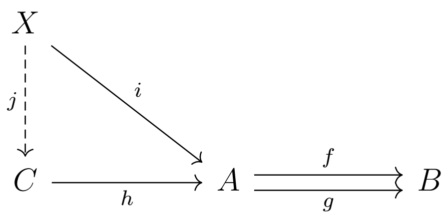
1 Category Theory (continued)

1.1 Baby Erlangen extended

Definition 1.1. Let $f, g : A \to B$ be two morphisms. Define the functor $Eq_{f,g} : \mathcal{C}^{op} \to \mathbf{Set}$ by $Eq_{f,g}(X) = \{i : X \to A \mid fi = gi\}$ and $Eq_{f,g}(j) = (-) \circ j$. By the equalizer of f and g, we mean the object C together with the natural isomorphism $Hom(X, C) \cong Eq_{f,g}(X)$. Equivalently, the equalizer of f and g is the object C together with a map $h : C \to A$ such that $fh = gh$, i.e.,



and for any other map $i : X \to A$ such that $fi = gi$, there exists a unique map $j : X \to C$ such that



It is called the equalizer of f and g, as it equalizes f and g.

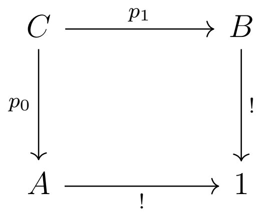
Example 1.2. In a poset as there is at most one map between any two objects, the equalizer of any pair $f, g : A \to B$ exists and it is $id_A : A \to A$. In any groupoid, any two maps $f, g : A \to B$ has the equalizer iff they are equal and the equalizer is again $id_A : A \to A$. More generally, the equalizer of two equal maps always exists and it is the identity of the source object.

Example 1.3. In **Set**, any two maps have the equalizer. Let $f, g : A \to B$ be two functions. It is easy to see that the set $C = \{x \in A \mid f(x) = g(x)\}$ together with the inclusion $i : C \to A$ is the equalizer. The same also works for **Grp**, **Ab** and $\mathbf{Vec}_{\mathbb{R}}$, in which C inherits the algebraic structure of A. For **Grp**, note that the equalizer of $f : G \to H$ and the constant map $c_e : G \to H$ mapping everything to e_H is exactly the kernel of f. More generally, if a category has a zero object (when $0 \cong 1$), then the kernel of a map $f : A \to B$ may be defined as the equalizer of f and $0_{A,B} : A \to 1 \cong 0 \to B$, where the maps $A \to 1$ and $0 \to B$ are the unique maps provided by the universal properties of 0 and 1.

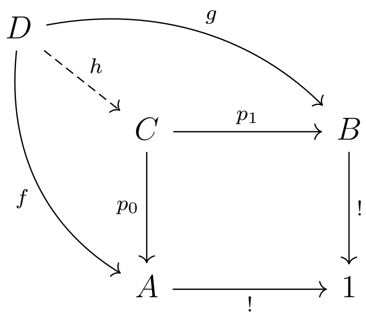
Example 1.4. In $\mathbf{Set}^{\mathcal{C}^{op}}$ any two maps have the equalizer and it is computed pointwise. Let $\alpha, \beta : F \Rightarrow G$ be two natural transformations. Define the functor $H : \mathcal{C}^{op} \to \mathbf{Set}$ on objects by $H(A) = \{x \in F(A) \mid \alpha_A(x) = \beta_A(x)\}$ and on morphism $f : B \to A$ by $H(f) = F(f)|_{H(A)} : H(A) \to H(B)$. It is easy to check that H is a functor, the canonical inclusion $i_A : H(A) \to F(A)$ is a natural transformation and the whole data is the equalizer of α and β.

Theorem 1.5. *Let $\mathcal{C}$ be a category that has the terminal object. Then, $\mathcal{C}$ has all pullbacks iff it has all binary products and all equalizers.*

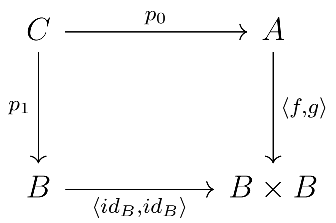
Proof. If a category has the terminal object and all pullbacks, then it has the binary product, computed as the pullback:



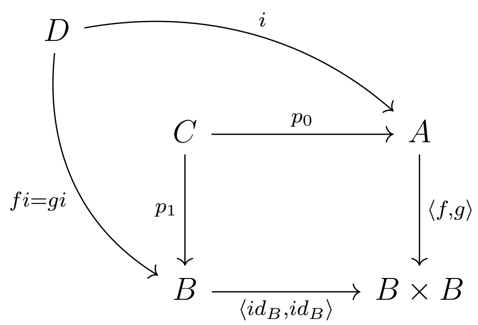
To prove that C, p_0 and p_1 is the product, note that if we have $f : D \to A$ and $g : D \to B$, then as there is only one map from D to 1, we have $!f =!g$, and as the square is a pullback, there exists a unique map $h : D \to C$ such that:



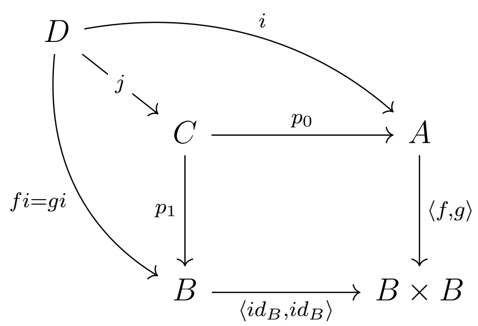
Now, we prove that all equalizers exist. Let $f, g : A \to B$ be two maps. Consider the following pullback:



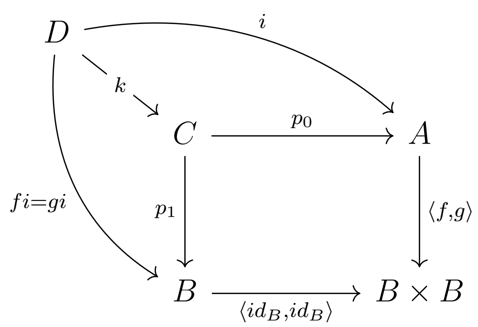
We claim that $p_0 : C \to A$ is the equalizer. First, as the square is commutative, we have $fp_0 = gp_0$. Moreover, if there is a map $i : D \to A$ such that $fi = gi$, then we have



As the square is a pullback, there is a map $j : D \to C$ such that $p_0 j = i$, i.e.,



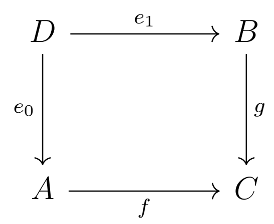
The only thing remains to prove it the uniqueness of this j. If there is $k : D \to C$ such that $p_0 k = i$, then it is easy to see that k we have:



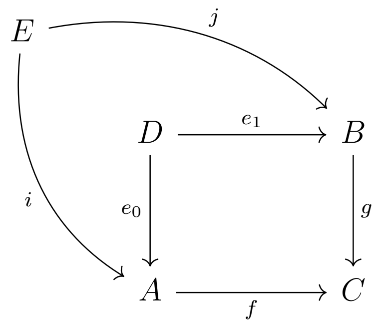
and as the square is the pullback, we have $k = j$.
Conversely, if the binary products and the equalizers exist, then pullback also exists. Let $f : A \to C$ and $g : B \to C$ be two maps. Then, consider the equalizer:

$$D \xrightarrow{e=\langle e_0, e_1 \rangle} A \times B \underset{gp_1}{\overset{fp_0}{\rightrightarrows}} C$$

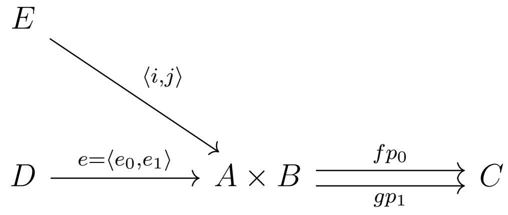
we claim that the diagram



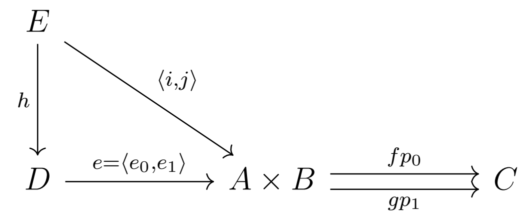
is a pullback. It is clearly commutative. To show the universality, if there is $i : E \to A$ and $j : E \to B$ such that



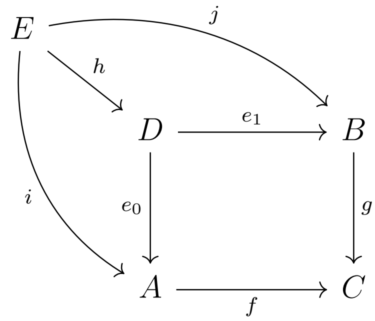
Then, we have



which by the fact that $e : D \to A \times B$ is equalizer, there exists a map $h : E \to D$ such that

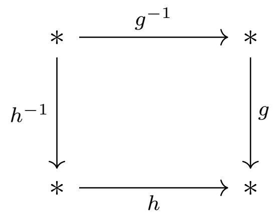


which implies

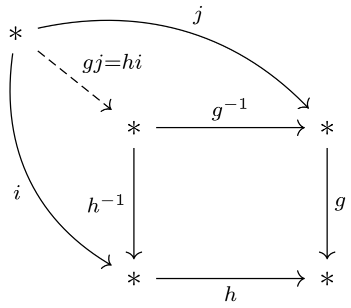


The uniqueness of $h : E \to D$ is easy. $\square$

Example 1.6. In the previous theorem, the existence of the terminal object is essential. For instance, if G is a non-trivial group as we observed before, if $g \neq h$, then they do not have equalizer. But all pullbacks in this category exist. The reason simply is that for any elements $g, h \in G$, the square

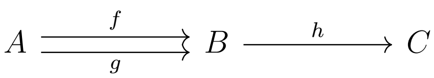


is a pullback, because it commutes and for any other $i, j \in G$ such that $gj = hi$, we have



The map $gj = hi$ is clearly unique.

Definition 1.7. Let $f, g : A \to B$ be two morphisms. Define the functor $CoEq_{f,g} : \mathcal{C} \to \mathbf{Set}$ by $CoEq_{f,g}(X) = \{i : B \to X \mid if = ig\}$ and $CoEq_{f,g}(j) = j \circ (-)$. By the coequalizer of f and g, we mean the object C together with the natural isomorphism $Hom(C, X) \cong CoEq_{f,g}(X)$. Equivalently, the coequalizer of f and g is the object C together with a map $h : B \to C$ such that $hf = hg$, i.e.,



and for any other map $i : B \to X$ such that $if = ig$, there exists a unique map $j : C \to X$ such that $jh = i$, i.e.,

$$\begin{array}{ccccc} A & \overset{f}{\underset{g}{\rightrightarrows}} & B & \xrightarrow{h} & C \\ & & & \searrow^{i} & \downarrow{\scriptstyle j} \\ & & & & X \end{array}$$

It is called the coequalizer as it is the dual of the equalizer.

Example 1.8. In a poset the coequalizer of any pair $f, g : A \to B$ exists and it is $id_B : B \to B$. In any groupoid, any two maps $f, g : A \to B$ has the coequalizer iff they are equal and the coequalizer is again $id_B : B \to B$. More generally, the coequalizer of two equal maps always exists and it is the identity of the target object. In **Set**, any two maps have the coequalizer. Let $f, g : A \to B$ be two functions. It is easy to see that the set $C = B/\sim$ together with the canonical projection $p : B \to C$ mapping b to $[b]$ is the coequalizer, where $\sim \subseteq B \times B$ is the least equivalence relation extending $\{(b, c) \in B \times B \mid \exists a \in A \; b = f(a) \text{ and } c = g(a)\}$. More specifically, if $R \subseteq B \times B$ is an equivalence relation, then B/R is just the coequalizer of $p_0, p_1 : R \to B$, where p_0 and p_1 are the projections. In **Top** the same construction works, except that we need the quotient topology. For instance, the coequalizer of the two ends of the interval $[0, 1]$ is $\mathbb{S}^1$:

$$\{0\} \overset{0 \mapsto 1}{\underset{0 \mapsto 0}{\rightrightarrows}} [0,1] \longrightarrow \mathbb{S}^1$$

For **Ab**, the coequalizer of $f, g : G \to H$ is the group $H/Im(f - g)$. Note that the cokernel of $f : G \to H$, i.e., $H/Im(f)$ is the coequalizer of f and $0 : G \to H$, where 0 is the map that sends everything to 0_H. More generally, if a category has a zero object, then the cokernel of a map $f : A \to B$ may be defined as the coequalizer of f and $0_{A,B} : A \to 1 \cong 0 \to B$.