Mathematical Structuralism, S19

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1 Category Theory (continued)

1.1 Baby Erlangen extended

Example 1.1. In a poset the coequalizer of any pair $f, g : A \to B$ exists and it is $id_B : B \to B$. In any groupoid, any two maps $f, g : A \to B$ has the coequalizer iff they are equal and the coequalizer is again $id_B : B \to B$. More generally, the coequalizer of two equal maps always exists and it is the identity of the target object. In **Set**, any two maps have the coequalizer. Let $f, g : A \to B$ be two functions. It is easy to see that the set $C = B/\sim$ together with the canonical projection $p : B \to C$ mapping b to $[b]$ is the coequalizer, where $\sim\, \subseteq B \times B$ is the least equivalence relation extending $\{(b, c) \in B \times B \mid \exists a \in A \; b = f(a) \text{ and } c = g(a)\}$. More specifically, if $R \subseteq B \times B$ is an equivalence relation, then B/R is just the coequalizer of $p_0, p_1 : R \to B$, where p_0 and p_1 are the projections. In **Top** the same construction works, except that we need the quotient topology. For instance, the coequalizer of the two ends of the interval $[0, 1]$ is $\mathbb{S}^1$:

$$\{0\} \underset{0 \mapsto 0}{\overset{0 \mapsto 1}{\rightrightarrows}} [0,1] \longrightarrow \mathbb{S}^1$$

For **Ab**, the coequalizer of $f, g : G \to H$ is the group $H/Im(f - g)$. Note that the cokernel of $f : G \to H$, i.e., $H/Im(f)$ is the coequalizer of f and $0 : G \to H$, where 0 is the map that sends everything to 0_H. More generally, if a category has a zero object, then the cokernel of a map $f : A \to B$ may be defined as the coequalizer of f and $0_{A,B} : A \to 1 \cong 0 \to B$.

Example 1.2. In **Cat**, the coequalizer of the functors $F, G : \mathbf{1} \to \mathbf{2}$ mapping the only object of $\mathbf{1}$ to the objects of $\mathbf{2}$ is the category $(\mathbb{N}, +)$ and the map

$P : \mathbf{2} \to (\mathbb{N}, +)$, mapping objects to the only object of $(\mathbb{N}, +)$ and the only non-trivial map of $\mathbf{2}$ to the map $1 \in \mathbb{N}$:

$$\mathbf{1} \underset{* \mapsto *}{\overset{* \mapsto \dagger}{\rightrightarrows}} \mathbf{2} \longrightarrow (\mathbb{N}, +)$$

Similarly, for canonical functors $F, G : \mathbf{1} \to \mathcal{I}$, where

$$\mathcal{I} : \qquad * \rightleftarrows \dagger$$

the coequalizer is $(\mathbb{Z}, +)$ together with the map $Q : \mathcal{I} \to (\mathbb{Z}, +)$, mapping the objects to the only object of $(\mathbb{Z}, +)$ and the two non-trivial maps of $\mathcal{I}$ to 1 and -1:

$$\mathbf{1} \underset{* \mapsto *}{\overset{* \mapsto \dagger}{\rightrightarrows}} \mathcal{I} \longrightarrow (\mathbb{Z}, +)$$

Reading $\mathcal{I}$ as the categorical version of the topological space $[0, 1]$, this coequalizer in **Cat** is reminiscent of the coequalizer

$$\{0\} \underset{0 \mapsto 0}{\overset{0 \mapsto 1}{\rightrightarrows}} [0, 1] \longrightarrow \mathbb{S}^1$$

in **Top**. Can we conclude that $(\mathbb{Z}, +)$ is the categorical version of the circle $\mathbb{S}^1$? Does it related to the fact that the fundamental group of $\mathbb{S}^1$ is $(\mathbb{Z}, +)$?

Example 1.3. In $\mathbf{Set}^{\mathcal{C}^{op}}$ any two maps has the coequalizer and it is computed pointwise. More precisely, let $\alpha, \beta : F \Rightarrow G$ be two natural transformations. It is easy to see that the the functor H defined by $H(A) = G(A)/R(A)$, where $R(A)$ is the least equivalence relation extending $\{(x, y) \in G(A) \mid \exists z \in F(A)\ \alpha_A(z) = x \text{ anbd } \beta_A(z) = y\}$ and for any $f : B \to A$ by $H(f) : H(A) \to H(B)$ as the canonical map induced by $G(f)$. It is easy to check that this H is a functor, the natural projection $p_A : G(A) \to H(A)$ is a natural transformation and the whole data is the coequalizer of α and β.

Philosophical Note 1.4. (*The Duality Principle*) Let ϕ be a statement about a category, purely written in the language of objects, arrows, identity and composition, using identity, boolean operations and quantifiers over objects and morphisms. We are also allowed to use parameter, meaning names for some given objects and arrows. For instance, the fact that $p_0 : C \to A$ and $p_1 : C \to B$ is the product of A and B is written as:

$$\forall f : D \to A\ \forall g : D \to B\ \exists!\ h : D \to C\ [(p_0 \circ h = f) \wedge (p_1 \circ h = g)]$$

with parameters $p_0 : C \to A$ and $p_1 : C \to B$. Then, by the dual statement of ϕ, denoted by ϕ^{op}, we mean the result of flipping all the arrows in ϕ and then changing $f \circ g$ by $g \circ f$, everywhere including in the parameters. For instance, the dual of the above statement is:

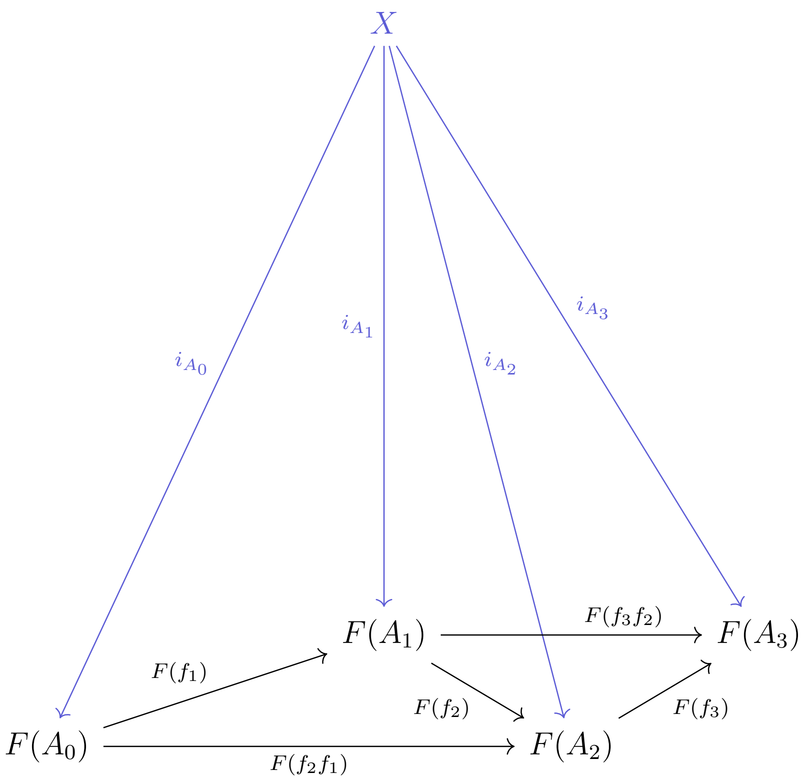
$$\forall f : A \to D\ \forall g : B \to D\ \exists!\ h : C \to D\ [(h \circ p_0 = f) \wedge (h \circ p_1 = g)]$$

for parameters $p_0 : A \to C$ and $p_1 : B \to C$. It is clear that the statement ϕ is true in $\mathcal{C}$ iff ϕ^{op} is true in $\mathcal{C}^{op}$. Now, as the opposite of any category is also a category, it is clear that if a statement ϕ is true in all categories, its dual also holds for all categories. Why?

Theorem 1.5. *Let $\mathcal{C}$ be a category that has the initial object. Then, $\mathcal{C}$ has all pushouts iff it has all binary coproducts and all coequalizers.*

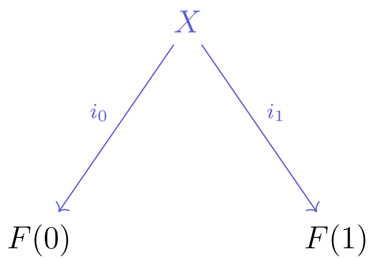
Proof. Use the duality principle. □

Now, we are ready to address the general case of limits. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (functor). Define a *cone over* F with the summit X as a natural transformation $\alpha : \Delta_X \Rightarrow F$. Spelling out, a cone over F with the summit X is an assignment $\{i_A : X \to F(A)\}_{A \in \mathcal{J}}$ such that $F(f)h_A = h_B$, for any $f : A \to B$, i.e.,

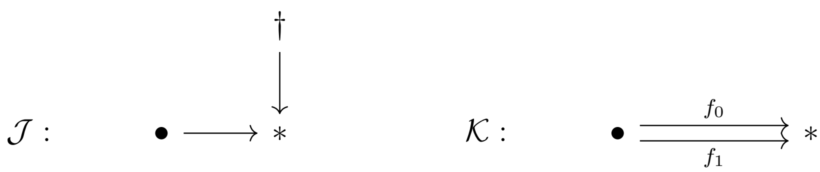


By the cone functor over F, we mean the functor $Cone_F : \mathcal{C}^{op} \to \mathbf{Set}$ defined by $Cone_F(X)$ as the set of all cones over F with summit X and for a map $j : B \to A$ by $Con_F(j) = j \circ (-)$.

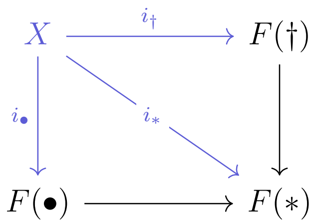
Example 1.6. Let $F : \mathbf{0} \to \mathcal{C}$ be the functor from the empty category to $\mathcal{C}$. Then, for any object X, there is exactly one cone over F with the summit X and hence $Cone_F = \Delta_{\{0\}}$. For any functor $F : \mathbf{1}+\mathbf{1} \to \mathcal{C}$, a cone over F with summit X is just the pair of two maps $f_0 : X \to F(0)$ and $f_1 : X \to F(1)$:



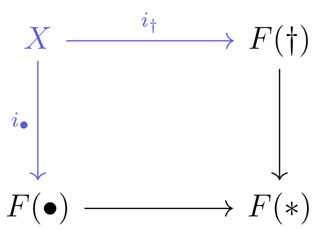
For more examples, define the following categories:



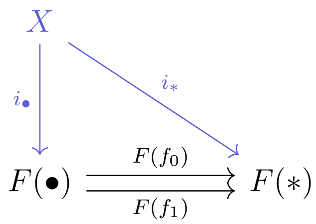
Then a cone over $F : \mathcal{J} \to \mathcal{C}$ with summit X is the tuple of three maps i_*, $i_\bullet$ and $i_\dagger$, such that:



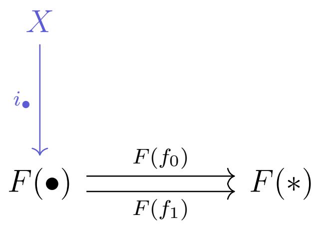
It is easy to see that the map i_* is uniquely determined by the maps $i_\bullet$ and $i_\dagger$ and hence there is no need to keep its data. Therefore, w.l.o.g, we can say that a cone over F with summit X is a pair of maps $i_\bullet$ and $i_\dagger$, such that:



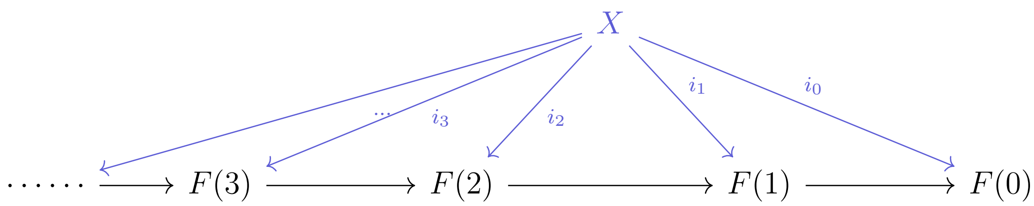
For a functor $F : \mathcal{K} \to \mathcal{C}$, a cone with summit X is a pair of maps $i : \bullet$ and i_* such that$i_* = F(f_0)i_\bullet$ and $i_* = F(f_1)i_\bullet$, i.e.,



Again, it is easy to see that i_* is uniquely determined by $i_\bullet$. However, this does not mean that we can pick any $i_\bullet$ as we want. The necessary and sufficient condition for $i_\bullet$ is that $F(f_0)i_\bullet = F(f_1)i_\bullet$:



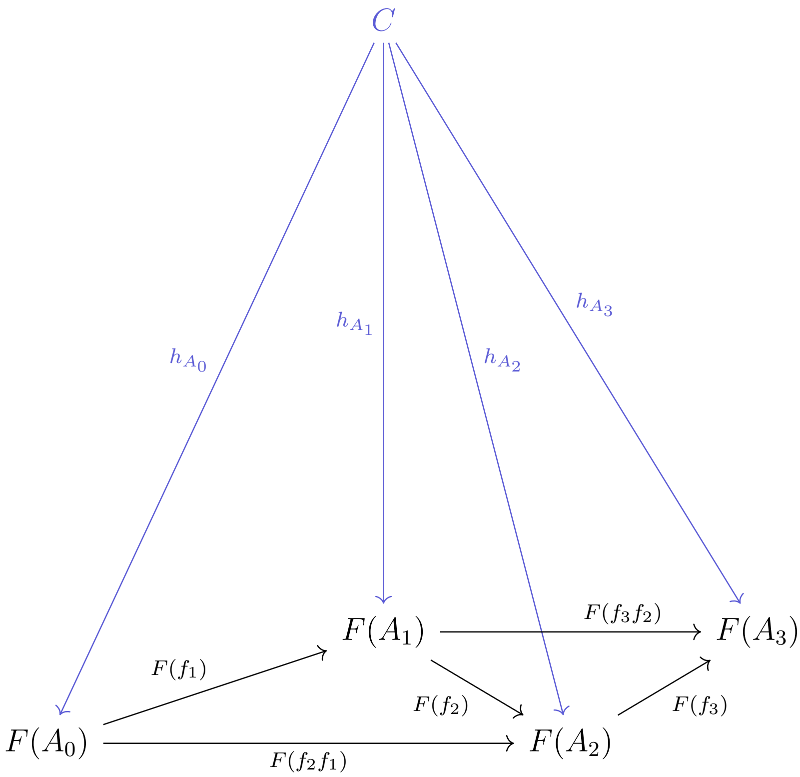
Therefore, w.l.o.g, we can say that a cone over F with summit X is a map $i_\bullet$ such that $F(f_0)i_\bullet = F(f_1)i_\bullet$.
For a functor $F : (\mathbb{N}, \leq)^{op} \to \mathcal{C}$, a cone with summit X is a sequence of maps $\{i_n : X \to F(n)\}_{n \in \mathbb{N}}$ such that:



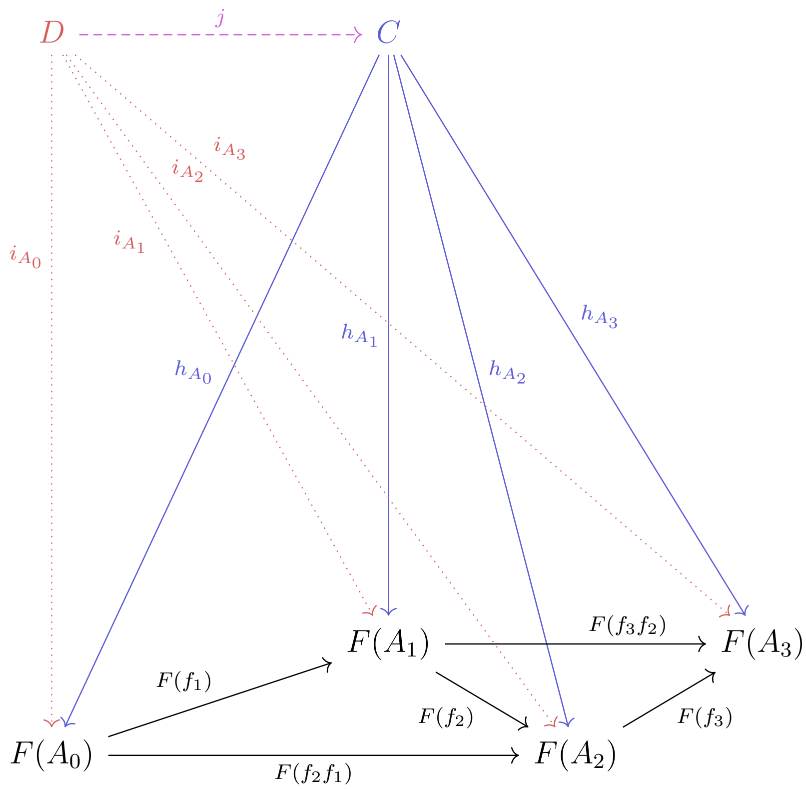
Can you explain why a cone over $F : (\mathbb{N}, \leq) \to \mathcal{C}$ is not interesting?

Definition 1.7. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (functor). By the limit of F, we mean an object C together with a natural isomorphism $Hom(X, C) \cong Cone_F(X)$. Equivalently, the limit of F is the object C together with a map $h_A : C \to F(A)$, for any object A in $\mathcal{J}$, such that $F(f)h_A = h_B$, for any

$f : A \to B$, i.e.,

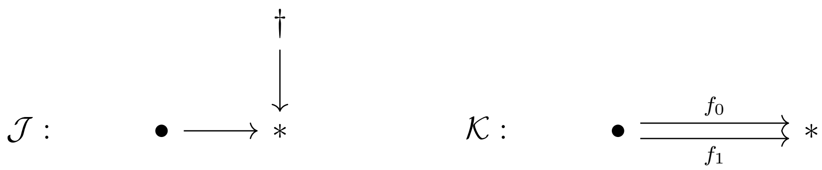


and for any other maps $i_A : D \to F(A)$, for any object A in $\mathcal{J}$ such that $F(f)i_A = i_B$, for any $f : A \to B$, there exists a unique map $j : D \to C$ such that $h_A j = i_A$, for any A, i.e.,



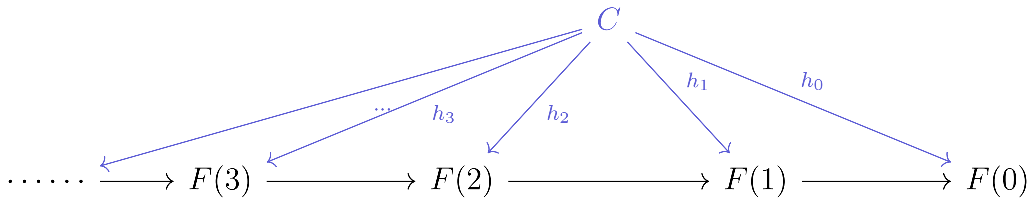
A limit is called (finite) small if the category $\mathcal{J}$ is (finite) small. A category is called complete, if it has all small limits and finitely complete, if it has all finite limits.

Example 1.8. Let $F : \mathbf{0} \to \mathcal{C}$ be the functor from the empty category to $\mathcal{C}$. The limit of F is the terminal object. For any functor $F : \mathbf{1} + \mathbf{1} \to \mathcal{C}$ the limit is the product of the objects in the image of F. Recall that we had the following categories:



Then, the limit of any functor $F : \mathcal{J} \to \mathcal{C}$ is the pullback of the F-image of the arrows along each other and the limit of any functor $F : \mathcal{K} \to \mathcal{C}$ is the equalizer of the F-image of the two arrows $F(f_0)$ and $F(f_1)$.

Example 1.9. Let $F : (\mathbb{N}, \leq)^{op} \to \mathcal{C}$ be a functor. The limit of F is called the *inverse limit* of the family $\{F(n+1 \geq n) : F(n+1) \to F(n)\}_{n\in\mathbb{N}}^{\infty}$:



Philosophical Note 1.10. (*Completion of Rings*) It is usually helpful to interpret a commutative unital ring R as a ring of some sore of "acceptable" functions from a "space" X to a fixed field F. For instance, we may think of the ring $\mathbb{C}[z]$ as a ring of polynomial functions from the space $\mathbb{C}$ to the field $\mathbb{C}$. Note that in this interpretation, we have no access to the space itself. We know the space through the quantities (functions) we can measure over it and hence we must reconstruct any property of the space from the ring, if it is possible. For instance, a "point" of the "space" may be identified by all the functions that vanish on the point and as F is supposed to be a field, the set of such functions forms a maximal ideal M. Hence, a "point" will be simply a maximal ideal of the ring R. For instance, in our above example, the point $0 \in \mathbb{C}$ is identified by the maximal ideal $\{r \in \mathbb{C}[z] \mid r(0) = 0\}$. Now, what is the value of the function $r \in R$ in the point M? Reading the value r_0 as a constant function, we expect that $r - r_0$ vanishes in the point M. Hence, $r - r_0 \in M$. As r_0 is invariant under any addition of functions that vanishes in M, it is reasonable to set r_0 as the remainder of r modulo M or $r + M \in R/M$. Note that with a similar argument, we can talk about the polynomial approximation of r around M with degree n as the remainder of r modulo M^n or $r + M^n \in R/M^n$.
Now, note that the ring of functions around a point can be incomplete in the sense that we may have a "convergent" sequence of functions whose *limit* does not exist in the original ring R. For instance, think about the sequence of polynomial approximations $\{\Sigma_{i=0}^{n} z^n/n!\}_{n=0}^{\infty}$ of the function e^z around the point $p = 0$. Is it possible to perform such a completion pure algebraically to reach a ring of "analytic functions" around a point? Let's give it a try! An analytic function, what it means, leaves a trace of polynomial approximations in our given ring R exactly as what the elements in R does. The value of the function is stored in R/M, the linear approximation lives in R/M^2 and so on. So any analytic function left the trace of a sequence $\langle r_n + M^n \rangle_n \in \{R/M^n\}_{n=0}^{\infty}$ as its "polynomial" approximations. Note that this sequence must have the property that $p_n(r_{n+1}) = r_n$, where $p_n : R/M^{n+1} \to R/M^n$ is the canonical projection as we expect that by increasing the degree of the approximation,

the partial results remain consistent in their lower degrees. Now, as we believe that an entity is nothing but its behavior, we may identify the analytic functions around M as the ring of these consistent sequences, i.e.,

$$\{\langle r_n + M^n \rangle_n \in \{R/M^n\}_{n=0}^{\infty} \mid \forall n \; p_n(r_{n+1}) = r_n\}$$

How to construct such a ring in pure categorical terms? It is simply the limit of the following diagram:

$$\cdots\cdots \xrightarrow{p_3} R/M^3 \xrightarrow{p_2} R/M^2 \xrightarrow{p_1} R/M$$

A special case of such a situation is the familiar case of p-adic numbers. Let $R = \mathbb{Z}$ and $M = p\mathbb{Z}$. Then, the limit is the ring of p-adic numbers and hence we can interpret any p-adic number as an analytic function around the abstract point $p\mathbb{Z}$ in an abstract space.

Example 1.11. (*Solenoids*) Interpreting $\mathbb{S}^1$ as the topological version of the group $(\mathbb{Z}, +)$, we may introduce the topological version of p-adic numbers as the limit of:

$$\cdots\cdots \xrightarrow{(-)^p} \mathbb{S}^1 \xrightarrow{(-)^p} \mathbb{S}^1 \xrightarrow{(-)^p} \mathbb{S}^1$$

where, $(-)^p : \mathbb{S}^1 \to \mathbb{S}^1$ is mapping the point $(cos(\theta), sin(\theta))$ to $(cos(p\theta), sin(p\theta))$. The space is called the p-solenoid.