Mathematical Structuralism, S20

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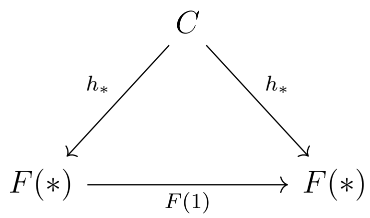
1 Category Theory (continued)

1.1 Baby Erlangen extended

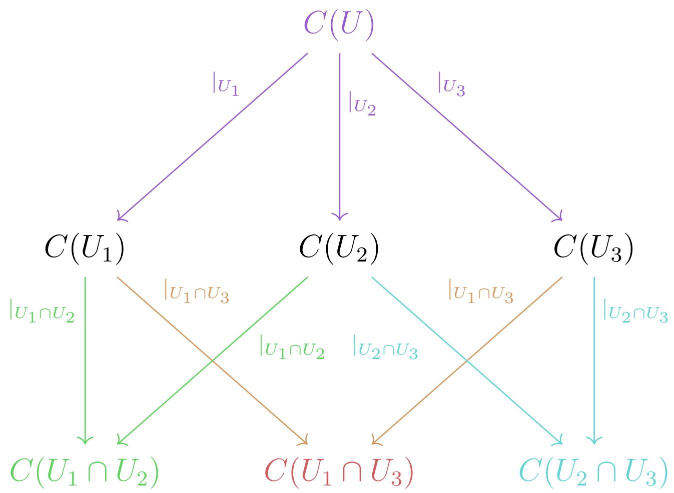
Example 1.1. If $\mathcal{J}$ is the monoid $(\mathbb{N}, +)$, the limit of a functor $F : (\mathbb{N}, +) \to$ **Set**



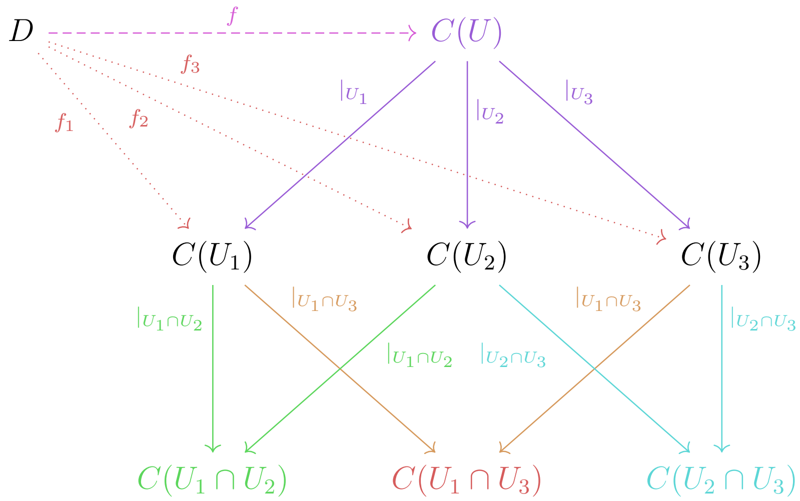
is just the set C of all fixed points of the function $F(1) : F(*) \to F(*)$ together with the inclusion map $h_* : C \to F(*)$.

Example 1.2. (*Sheaves*) Let X be a topological space, $\{U_i\}_{i \in I}$ be a family of open subsets and $U = \bigcup_{i \in I} U_i$. Define the functor $C : \mathcal{O}(X)^{op} \to$ **Set** on the open subset V of X by $C(V) = \{f : V \to \mathbb{R} \mid f \text{ is continuous}\}$ and on the unique morphism $V \supseteq W$ as the restriction map $|_W : C(V) \to C(W)$. The functor C stores all partial continuous functions over X defined on an open domain. Define P as the set of all $\{i, j\}$'s, where $i, j \in I$ and $F : (P, \subseteq) \to$ **Set** as the diagram mapping $\{i, j\}$ to $C(U_i \cap U_j)$ and the only non-trivial morphism $\{i\} \subseteq \{i, j\}$, for $i \neq j$ to the restriction map

$|_{U_i \cap U_j} : C(U_i) \to C(U_i \cap U_j)$. Then, $C(U)$ is the limit of the diagram F:



The diagram is clearly commutative. To show its universality, for any other cone $\{f_i : D \to C(U_i)\}_{i \in I}$:

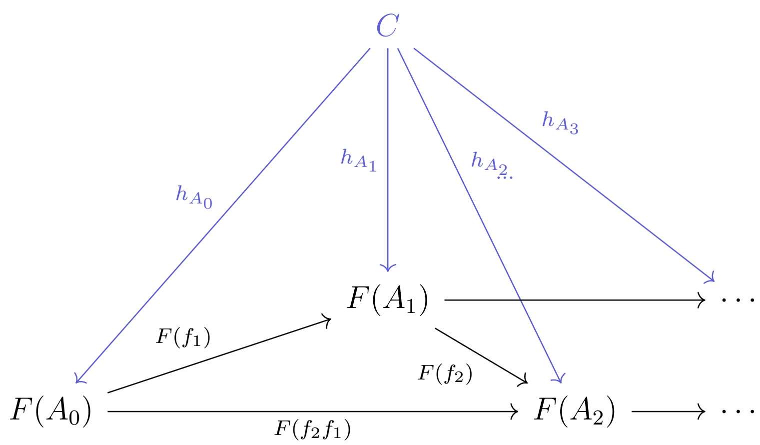


the commutativity of the diagram states that for any $x \in D$, the functions $\{f_i(x) : U_i \to \mathbb{R}\}_{i \in I}$ are consistent on the intersections of their domains and hence we can construct a unique function on U by their union. Set $f(x) : U \to \mathbb{R}$ as this unique function and note that f is continuous. It is easy to see that this f is the only map we can use here.
Note that the main reason behind the argument is that the fact that continuity is a local notion meaning that continuity in a point is determined by the behavior of the function on a small neighbourhood of x. This implies that if we have a consistent family of continuous functions on some opens, we

can glue them together to construct a unique continuous function extending them all. Changing continuity to any other local notion like derivability also works while using global notions like constancy breaks the argument.
To formalize the general situation, let $G : \mathcal{O}(X)^{op} \to \mathbf{Set}$ be a functor. If for any family of opens $\{U_i\}_{i \in I}$ covering U, the set $G(U)$ is the limit of the corresponding functor F, we call G a sheaf over X. We can think of a *sheaf* as a machine to store all the *local* instance of a *local* notion.

Example 1.3. A poset is (finitely) complete iff it has all (finite) meets. One direction is clear. For the other direction, let $F : \mathcal{J} \to (P, \leq)$ be a diagram. Then, as in a poset any two maps with the same source and target are equal, we can observe that the meet $\bigwedge_J F(J)$ together with its unique map to all $F(J)$'s is the limit of F. For instance, the poset $(P(X), \subseteq)$ is complete as it has all possible meets.

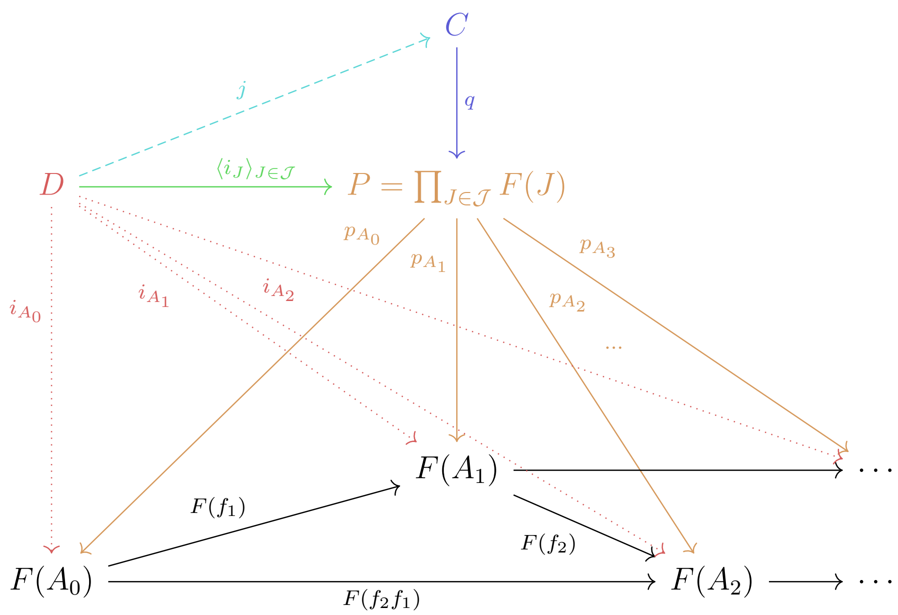
Example 1.4. The category $\mathbf{Set}$ is complete. To prove that, let $F : \mathcal{J} \to \mathbf{Set}$ be a small diagram. Then, define $C = \{s \in \prod_{A \in \mathcal{J}} F(A) \mid \forall f : A \to B\ [F(f)(s(A)) = s(B)\}$ and $h_A : C \to F(A)$ as the canonical projection on A'th element:



It is easy to see that this data is the limit of F. The same construction with the pointwise algebraic structure also works for **Grp**, **Ab** and $\mathbf{Vec}_{\mathbb{R}}$. For **Top**, we also have the same construction, this time using the product and the subspace topology. Note that the subcategory of all compact Hausdorff spaces is also complete. The reason is simply the combination of the Tychonoff's theorem and the fact that the subspace defined by any number of equalities is compact.

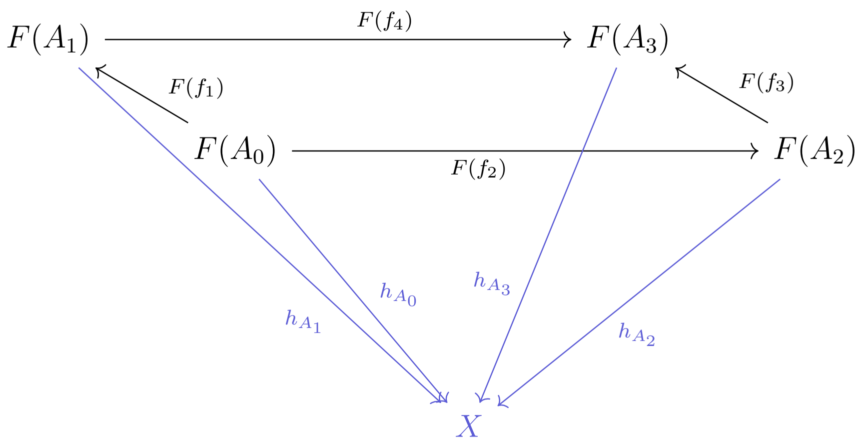
Theorem 1.5. *A category is (finitely) complete iff it has all (finite) products and all equalizers.*

Proof. One direction is clear by definition. For the other direction, we use the argument from the previous example. Let $F : \mathcal{J} \to \mathbf{Set}$ be a small diagram. Then, as products of size of $\mathcal{J}$ exists, the product $\prod_{J \in \mathcal{J}} F(J)$ with projections $p_J : P \to F(J)$ exists. Set $P = \prod_{J \in \mathcal{J}} F(J)$. Then, set C and $q : C \to P$ as the equalizer of $\langle p_K \rangle_{J,f}, \langle F(f) p_J \rangle_{J,f} : \prod_{J \in \mathcal{J}} F(J) \to \prod_{J \in \mathcal{J}} \prod_{f:J \to K} F(K)$. The limit will be $\{p_J q : C \to F(J)\}_{J \in \mathcal{J}}$. As $q : C \to P$ is the equalizer, we have $p_K q = F(f) p_J q$. To show that it is the best choice, assume $\{i_J : D \to F(J)\}_{J \in \mathcal{J}}$ has the property $i_K = F(f) i_J$, for any $f : J \to K$:



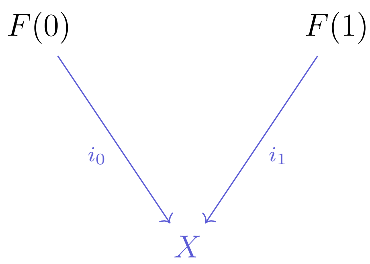
Therefore, $\langle p_K \rangle_{J,f} \circ \langle i_J \rangle_J = \langle F(f) p_J \rangle_{J,f} \circ \langle i_J \rangle_J$. As $q : C \to P$ is the equalizer, there exists a unique map $j : D \to C$ such that $qj = \langle i_J \rangle_J$. Hence, $p_J q j = i_J$. Uniqueness condition for limit is also easy. □

Now, let us spell out the dual notion of *cones under a diagram* or *cocones* and *colimits*. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (functor). Define a *cone under* F with the nadir X as a natural transformation $\alpha : F \Rightarrow \Delta_X$. Spelling out, a cone under F with the nadir X is an assignment $\{i_A : F(A) \to X\}_{A \in \mathcal{J}}$ such that $h_A = h_B F(f)$, for any $f : A \to B$, i.e.,

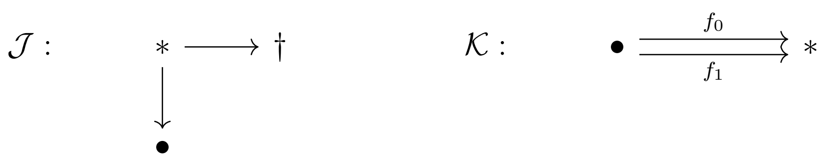


By the cone functor under F, we mean the functor $Cone^F : \mathcal{C} \to \mathbf{Set}$ defined by $Cone^F(X)$ as the set of all cones under F with nadir X and for a map $j : A \to B$ by $Con^F(j) = (-) \circ j$.

Example 1.6. Let $F : \mathbf{0} \to \mathcal{C}$ be the functor from the empty category to $\mathcal{C}$. Then, for any object X, there is exactly one cone under F with the nadir X and hence $Cone^F = \Delta_{\{0\}}$. For any functor $F : \mathbf{1} + \mathbf{1} \to \mathcal{C}$, a cone under F with nadir X is just the pair of two maps $i_0 : F(0) \to X$ and $i_1 : F(1) \to X$:

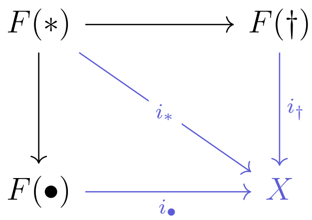


For more examples, define the following categories:

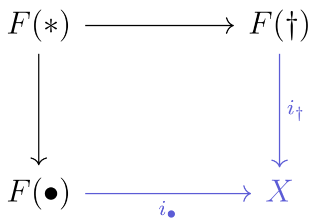


Then a cone under $F : \mathcal{J} \to \mathcal{C}$ with nadir X is the tuple of three maps i_*, $i_\bullet$

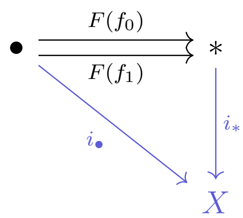
and $i_\dagger$, such that:



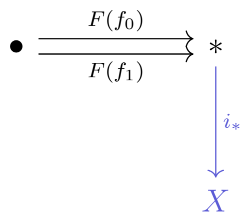
It is easy to see that the map i_* is uniquely determined by the maps $i_\bullet$ and $i_\dagger$ and hence there is no need to keep its data. Therefore, w.l.o.g, we can say that a cone under F with nadir X is a pair of maps $i_\bullet$ and $i_\dagger$, such that:



For a functor $F : \mathcal{K} \to \mathcal{C}$, a cone under F with nadir X is a pair of maps $i_\bullet$ and i_* such that$i_\bullet = i_* F(f_0)$ and $i_\bullet = i_* F(f_1)$, i.e.,

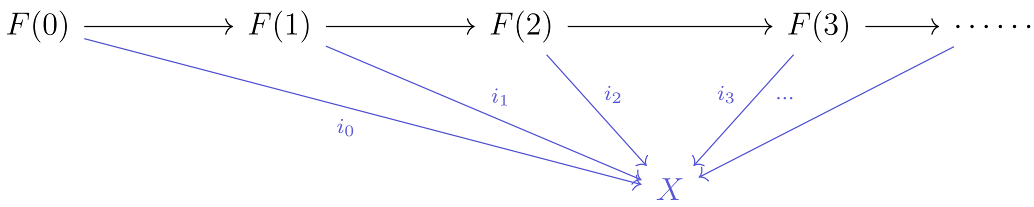


Again, it is easy to see that $i_\bullet$ is uniquely determined by i_*. However, this does not mean that we can pick any i_* as we want. The necessary and sufficient condition for i_* is that $i_* F(f_0) = i_* F(f_1)$:



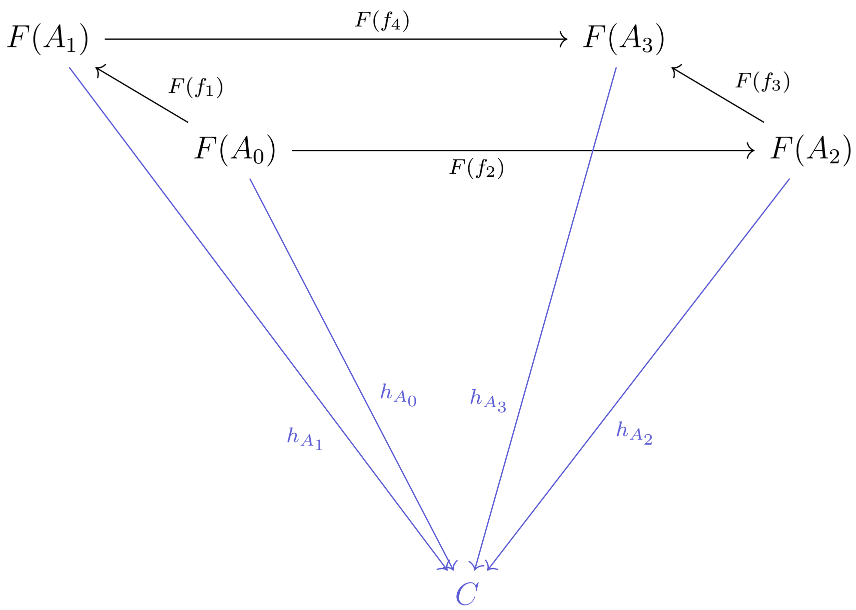
Therefore, w.l.o.g, we can say that a cone under F with nadir X is a map i_* such that $i_* F(f_0) = i_* F(f_1)$.
For a functor $F : (\mathbb{N}, \leq) \to \mathcal{C}$, a cone under F with nadir X is a sequence of

maps $\{i_n : F(n) \to X\}_{n\in\mathbb{N}}$ such that:

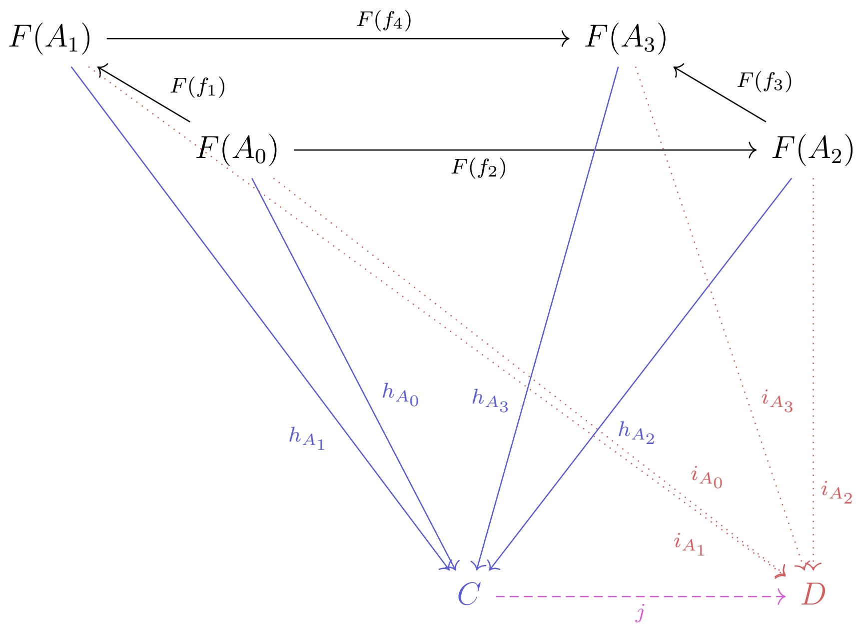


Can you explain why a cone under $F : (\mathbb{N}, \leq)^{op} \to \mathcal{C}$ is not interesting?

Definition 1.7. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram (functor). By the colimit of F, we mean an object C together with a natural isomorphism $Hom(C, X) \cong Cone^F(X)$. Equivalently, the colimit of F is the object C together with a map $h_A : F(A) \to C$, for any object A in $\mathcal{J}$, such that $h_A = h_B F(f)$, for any $f : A \to B$, i.e.,



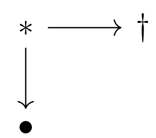
and for any other maps $i_A : F(A) \to D$, for any object A in $\mathcal{J}$ such that $F(f)i_A = i_B$, for any $f : A \to B$, there exists a unique map $j : C \to D$ such that $jh_A = i_A$, for any A, i.e.,



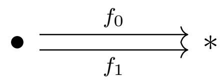
the category $\mathcal{J}$ is (finite) small. A category is called cocomplete, if it has all small colimits and finitely cocomplete, if it has all finite colimits.

Example 1.8. Let $F : \mathbf{0} \to \mathcal{C}$ be the functor from the empty category to $\mathcal{C}$. The colimit of F is the initial object. For any functor $F : \mathbf{1} + \mathbf{1} \to \mathcal{C}$ the limit is the coproduct of the objects in the image of F. Recall that we had the following categories:

$\mathcal{J}$:

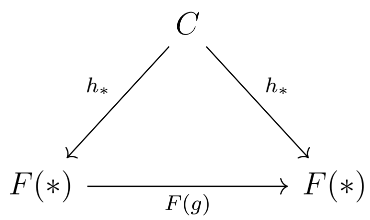


$\mathcal{K}$:



Then, the colimit of any functor $F : \mathcal{J} \to \mathcal{C}$ is the pushout of the F-image of the arrows along each other and the colimit of any functor $F : \mathcal{K} \to \mathcal{C}$ is the equalizer of the F-image of the two arrows $F(f_0)$ and $F(f_1)$.

Example 1.9. If $\mathcal{J}$ is a group G and $F : G \to \mathbf{Set}$ be a G-action. Then, the colimit of F:



is just the set $F(*)/R$ where $R = \{(x, y) \in F(*) \times F(*) \mid \exists g \in G\ F(g)(x) = y\}$ together with the projection map $h_* : F(*) \to F(*)/R$. The set $F(*)/R$ is actually the set of all orbits.

Example 1.10. Any group is a colimit of its finitely-generated subgroups. More formally, let G be a group and $\mathcal{J}$ be the poset of all finitely-generated subgroups of G with the inclusion. Then, if $F : \mathcal{J} \to \mathbf{Grp}$ is the inclusion functor, the colimit of F is G with legs $h_H : H \to G$ as the inclusion homomorphism. It is clear that the diagram is commutative. To show that G is the best choice, assume $\{i_H : H \to K\}_{H \in \mathcal{J}}$ be a cone under F. Then, define $j : G \to K$ by $j(g) = i_{\langle g \rangle}(g)$, where $\langle g \rangle$ is the cyclic group generated by $g \in G$. The map j is a homomorphism, i.e., $j(gg') = j(g)j(g')$. As $\{i_H : H \to K\}_{H \in \mathcal{J}}$ is a cone under F, we have $i_{\langle g \rangle}(g) = i_{\langle g, g' \rangle}(g)$. Similarly, we have $i_{\langle g' \rangle}(g') = i_{\langle g, g' \rangle}(g')$ and $i_{\langle gg' \rangle}(gg') = i_{\langle g, g' \rangle}(gg')$. Since $i_{\langle g, g' \rangle}$ is a homomorphism, we have $i_{\langle g, g' \rangle}(gg') = i_{\langle g, g' \rangle}(g) i_{\langle g, g' \rangle}(g')$. Hence, $i_{\langle gg' \rangle}(gg') = i_{\langle g \rangle}(g) i_{\langle g' \rangle}(g')$. The map i_H is the composition of the inclusion and the map j. The argument is again similar to what we did for proving that j is a homomorphism. The uniqueness of such j is obvious.

Philosophical Note 1.11. The main reason why the previous example works is twofold. First, the fact that we are working with algebras (sets equipped with some operators satisfying certain equations) and second that the operators of the algebras (in our example, the product) are finitary. For instance, to show that j preserves the operators, we need to put all the inputs of the operator in one finitely-generated algebra which needs the number of these inputs (the arity of the operator) to be finite. Philosophically speaking, we can say that in the finitary algebraic world (groups, rings, etc) we can *construct* an algebra by their finitely-generated subalgebras and hence understanding the maps *going out* from an algebra reduced to the maps *going out* from some finitely-generated algebras .