Mathematical Structuralism, S21

Amir Tabatabai

July 29, 2021

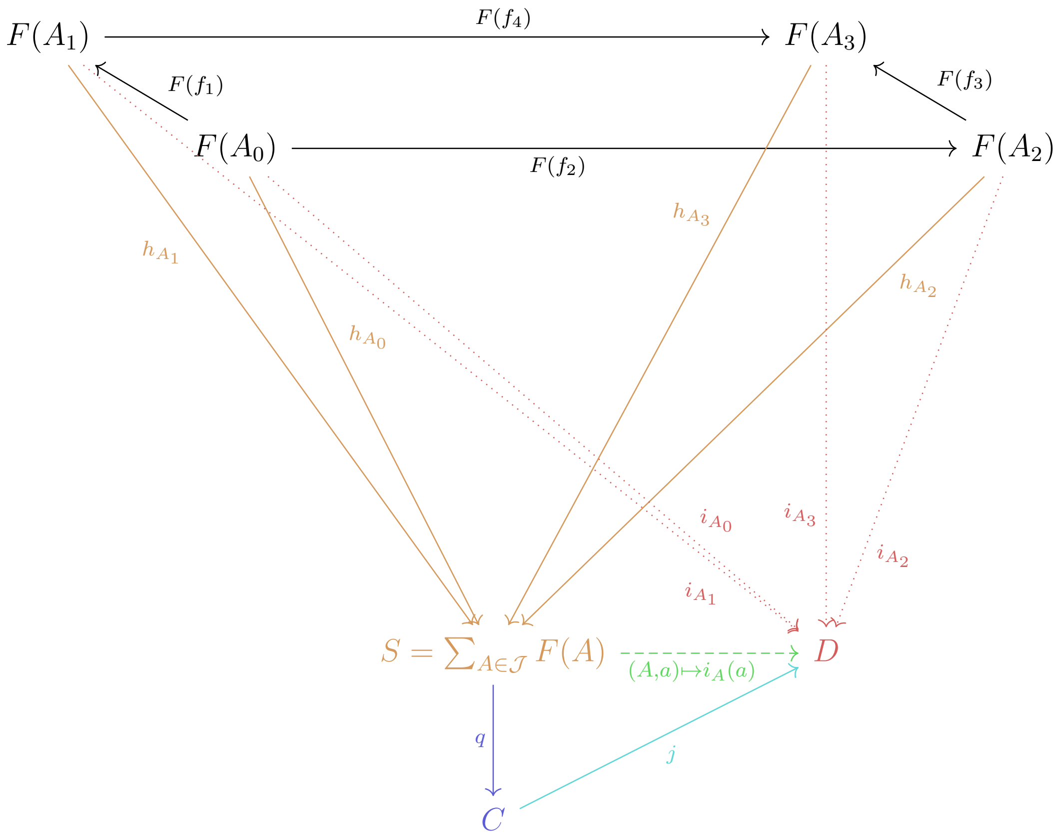
1 Category Theory (continued)

1.1 Baby Erlangen extended

Example 1.1. (*Germ of functions*) Let X and Y be topological spaces and $x \in X$ be a point. Assume that we are interested in the *local behavior* of partial continuous functions from X to Y defined on a neighbourhood around x. By local behavior, we mean the aspects of f that depends on its values on *a* neighbourhood of x. To capture these aspects, we need the *local equality*. We call two functions locally equal at x, if they are equal on *a* neighbourhood around x. This local equality is an equivalence relation. Call the equivalence class of a function its *germ at* x. It is clear that the germ of f at x captures the local aspect of x. For instance, think about the derivative of f at x, if it exists. This derivative actually depends on the germ of f not the f itself. The natural machinery to implement this "up to local equality" is colimits. More precisely, let $\mathcal{J}$ be the opposite of the poset of all opens around x with inclusion. Then, set $F : \mathcal{J} \to \mathbf{Set}$ as $F(U) = \{f : U \to Y \mid f \text{ is continuous.}\}$ and on $U \supseteq V$ as the restriction map $|_V : F(U) \to F(V)$. Then, the colimit of F, denoted by F_x, is a set of all germs of functions at x and the leg $F(U) \to F_x$ maps a function on U to its germ at x.

Example 1.2. A poset is (finitely) cocomplete iff it has all (finite) joins. One direction is clear. For the other direction, let $F : \mathcal{J} \to (P, \leq)$ be a diagram. Then, as in a poset any two maps with the same source and target are equal, we can observe that the join $\bigvee_J F(J)$ together with the unique maps from $F(J)$'s is the colimit of F. For instance, the poset $(P(X), \subseteq)$ is complete as it has all possible joins.

Example 1.3. The category **Set** is cocomplete. To prove that, let $F : \mathcal{J} \to \mathbf{Set}$ be a small diagram. Then, define $C = \sum_{A \in \mathcal{J}} F(A)/\sim$, where $\sum_{A \in \mathcal{J}} F(A) = \{(A, a) \mid A \in \mathcal{J}, a \in F(A)\}$ and $\sim$ is the smallest equivalence relation generated by $\{(A,a) \sim (B,b) \mid \exists f : A \to B\ F(f)(a) = b\}$ and $h_A : F(A) \to \sum_{A\in\mathcal{J}} F(A)$ as the canonical injection that maps $a \in F(A)$ to (A,a) and $q : \sum_{A\in\mathcal{J}} F(A) \to C$ as the canonical quotient map:



It is easy to see that the cone $\{qh_A : F(A) \to C\}_{A\in\mathcal{J}}$ is the colimit of F.

Theorem 1.4. *A category is (finitely) cocomplete iff it has all (finite) coproducts and all coequalizers.*

Proof. Use the duality principle and the similar theorem for limits. □

Corollary 1.5. *The categories* **Set**, **Grp**, **Ab**, *and* **Top** *are both complete and cocomplete.*

Theorem 1.6. *A poset is cocomplete iff it is complete.*

Proof. Using the duality principle, it is enough to show that if $(P,\leq)$ is complete, it is also cocomplete. Let $S \subseteq P$. To show that it has the join, set $a = \bigwedge T$ where $T = \{x \in P \mid \forall s \in S\, x \geq s\}$, i.e., as the meet of all the upper bounds of S. It exists as the poset is complete. It is clear that a is an upper bound of S. Because, for any $s \in S$ and $x \in T$, we have $s \leq x$ and hence $s \leq \bigwedge T = a$. To show that it is the least upper bound, assume z is an upper bound of S. Hence, $z \in T$ and then $a \leq z$. $\square$

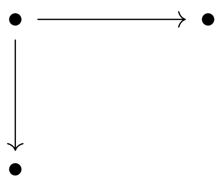
Example 1.7. Let $(C, \subseteq)$ be the class of all sets together with the inclusion as the order. This category is cocomplete as the union of any *set* of sets is a set but it is not even finitely complete as there is no maximum set containing all sets.

Exercise 1.8. Find a complete category that is not cocomplete.

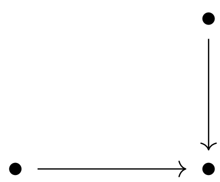
Definition 1.9. A small category $\mathcal{J}$ is called filtered if for any finite diagram $D : \mathcal{I} \to \mathcal{J}$, there is a cone under D. It is called cofiltered if for any finite diagram $D : \mathcal{I} \to \mathcal{J}$, there is a cone over D.

Example 1.10. A poset is filtered if any finite set has an upper bound. It is cofiltered iff any finite set has a lower bound. For instance, the poset $(Fin(X), \subseteq)$, where $Fin(X)$ is the set of all finite subsets of an infinite set X is filtered. Moreover, in a topological space X and for a subspace $A \subseteq X$, the poset $(N(A), \subseteq)$, where $N(A)$ is the set of all opens U containing A is cofiltered.

Example 1.11. Any small category with the terminal object or all finite colimits is filtered. As a non-example, the category



is not filtered. Dually, any small category with the initial object or all finite limits is cofiltered while the category



is not cofiltered.

Theorem 1.12. *A category is filtered iff the following three conditions hold:*

- *It has at least one object.*

- *For any two objects A and B, there is an object C and two morphisms $f : A \to C$ and $g : B \to C$.*
- *For any two maps $f, g : A \to B$, there is an object C and a morphism $h : B \to C$ such that $hf = hg$.*

Dually, a category is cofiltered iff the dual of the previous three conditions hold.

Proof. One directions is obvious. For the other direction, we use a similar argument as in Theorem **??**. □

Definition 1.13. The colimit of a digram $F : \mathcal{J} \to \mathcal{C}$ is called filtered if $\mathcal{J}$ is filtered. Dually, the limit of a digram $F : \mathcal{J} \to \mathcal{C}$ is called cofiltered if $\mathcal{J}$ is cofiltered.

Philosophical Note 1.14. Colimits can be interpreted as a way of constructing an object by "*gluing*" some "*simpler*" objects. In this sense, a filtered colimit is a way of construction where any divergence in the process of construction converges at some future step.

Example 1.15. As observed previously, any group is a colimit of its finitely generated subgroups. The colimit is actually filtered as the finite colimit of finitely generated groups is also finitely generated. The same type of argument also works for the rings and vector spaces.

Example 1.16. Let $F : \mathcal{O}(X)^{op} \to \mathbf{Set}$ be the functor mapping the open $U \in \mathcal{O}(X)$ to the set of the real-valued continuous functions on U. Then, if $x \in X$ is a point, the set F_x of the germs of the continuous functions at x is not only the colimit of $\{F(V)\}_{x \in V}$, but a filtered colimit as $(N(\{x\}), \subseteq)^{op}$ is a filtered category.

Example 1.17. Let $GL_n(\mathbb{R})$ be the group of all invertible $n \times n$ matrices and $i_n : GL_n(\mathbb{R}) \to GL_{n+1}(\mathbb{R})$ be the homomorphism that maps an $n \times n$ matrix A to the $(n+1) \times (n+1)$ matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Then, as **Grp** is cocomplete, the colimit of the diagram

$$\cdots \xrightarrow{i_{n-1}} GL_n(\mathbb{R}) \xrightarrow{i_n} GL_{n+1}(\mathbb{R}) \xrightarrow{i_{n+1}} \cdots$$

exists. This group is called $GL_\infty(\mathbb{R})$ and the colimit is clearly filtered as the index category $(\mathbb{N}, \leq)$ is filtered.