Universal Proof Theory: Almost Positive Rules and Feasible Disjunction Property

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Abstract

In this paper we will present a class of rules called the *almost positive rules* to show that any proof system for an intuitionistic modal logic that consists of theses rules, the cut and the necessitation rule has the feasible disjunction property. This method uniformly proves the property for the usual sequent-style and Hilbert-style proof systems for a broad range of intuitionistic modal logics, including IK, IKT, IK4, IS4, IS5, their Fisher-Servi versions, the intuitionistic logics for bounded depth and bounded width and the propositional lax logic. On the negative side, though, it shows that if an intuitionistic modal logic does not admit the Visser rules or specially does not have the disjunction property, then it does not have a calculus consisting only of almost positive rules, the cut rule and the necessitation rule. As the class of these rules is a general and natural class to consider, this negative result presents an interesting proof theoretical result about generic proof systems and their existence.

1 Introduction

It is well-known that the intuitionistic logic enjoys the disjunction property meaning that if $A \vee B$ is provable, then either A or B is provable. Gazing upon this problem from the computational complexity point of view, we can wonder about the complexity of the process that finds the proof of Aor B from a proof of $A \vee B$. Addressing this issue, Buss and Mints [6] and later Buss and Pudlák [7] showed that this process for propositional intuitionistic logic is polynomial-time or less formally feasible. More precisely, given a calculus $\mathbf{C}$ for the intuitionistic logic, there exists a polynomial time algorithm that reading a proof π of the formula $A \vee B$ in $\mathbf{C}$ outputs a proof of either A

or B in **C**. The calculus **C** considered in [6] was the natural deduction system and in [7] was Gentzen's sequent calculus. In both these papers, a form of normalization or cut elimination is needed. Later, Ferrari et al. ([8] and [9]) provided a uniform framework to study the complexity of the disjunction property in intuitionistic logic, some modal and intuitionistic modal logics. The method they used is based on a calculus called the extraction calculus. The benefit of their method, compared to the previous ones, is that they do not take the structural properties of the system **C** into account. Instead, their method uses an important distinction between the calculus **C** that the proof π is provided in and the extraction calculus in which the disjunction property is proved. For the intuitionistic modal logics, the disjunction property is defined as in intuitionistic logic. However, for modal logics, it implies the proof of A or B from a proof of $\Box A \vee \Box B$. The feasibility of disjunction property in several modal logics has been shown by Bílková [5] and for Frege systems for any extensible modal logic by Jeřábek [16].

Intuitionistic modal logics have been studied immensely due to their applications in various fields from philosophy and mathematical foundations to computer science. They are obtained by adding modalities to the intuitionistic logic. From another perspective, combining the well established classical modal logics and superintuitionistic logics can be used to build reasonable modal logics on an intuitionistic basis. Since the modalities $\Box$ and $\Diamond$ are not dual of each other in an intuitionistic setting, there are several ways of defining intuitionistic modal logics (see for instance [3], [4], [10], [12], [17], [19]). More on various intuitionistic modal logics is covered in Preliminaries. For a nice survey on intuitionistic modal logics see [18].

Recently, Iemhoff ([14], [15]) and later the authors of the current paper ([1], [2]) studied the relation between general forms of sequent calculi and the mathematical properties that the corresponding logic enjoys. Iemhoff showed that if the rules in a sequent calculus for a superintuitionistic logic are of a special form, then the corresponding logic enjoys uniform interpolation. Since there are only 7 superintuitionistic logics with uniform interpolation, she concluded that almost all superintuitionistic logics cannot have a sequent calculus of the mentioned form. Later, in [1] and [2] this result was strengthened to also cover the substructural (modal) logics. Moreover, the form of the rules was made more general and the Craig interpolation property was also studied. This was the birth of an approach to the study of proof system called universal proof theory. The present paper is also in line with the aforementioned research. We present a sequent calculus containing a special form of the rules and prove that any sequent calculus of this form enjoys feasible disjunction property. Therefore, any superintuitionistic logic that does not enjoy disjunction property cannot have a sequent calculus of the mentioned form.

The paper is organized as follows. We begin with some preliminaries in Section 2. Specially, we introduce a sequent calculus **iK** and we will use it as the basic case of our results. In Section 3 we introduce almost positive rules. These are the rules with a special form that we are interested in. In Section 4, we present our main result. We show that any sequent calculus **G** for an intuitionistic modal logic that is stronger than **iK** and only consists of almost positive rule has the feasible disjunction property. The result is in fact stronger. We prove that there exists a poly-time algorithm that reads a proof of $\Gamma, \{A_i \to B_i\}_{i \in I} \Rightarrow C \vee D$ in **G**, where Γ is a multiset of Harrop formulas and outputs a **G**-proof either for $\Gamma \Rightarrow C$ or $\Gamma \Rightarrow D$ or $\Gamma \Rightarrow A_i$, for some $i \in I$. The method we use to prove this result is inspired by the technique that Hrubeš used in [13] to prove an exponential lower bound on the lengths of proofs in the intuitionistic Frege system. The analogue of this result is provided in subsections 4.1, 4.2, and 4.3 for the $\Diamond$-free, $\Box$-free, and propositional fragments, respectively. In Section 5, the cut-free case is discussed. A certain form of rules, called **LJ**-like, and a form of axioms are introduced. Then we show that there is only one superintuitionistic logic that has a sequent calculus consisting of these axioms and rules, and that is the intuitionistic logic, IPC.

2 Preliminaries

In this paper, we mainly work with the language $\mathcal{L} = \{\wedge, \vee, \to, \top, \bot, \Box, \Diamond\}$. However, sometimes we also use the following three fragments of $\mathcal{L}$, i.e., $\mathcal{L}_\Box = \mathcal{L} \backslash \{\Diamond\}$, $\mathcal{L}_\Diamond = \mathcal{L} \backslash \{\Box\}$ and $\mathcal{L}_p = \mathcal{L} \backslash \{\Box, \Diamond\}$. Small Greek letters $\phi, \psi, \ldots$ and capital Roman letters $A, B, \ldots$, possibly with indices, denote formulas. Formulas in a language $\mathfrak{L}$ are also sometimes denoted as $\mathfrak{L}$-formulas. Capital Greek letters (possibly with indices) denote multisets of formulas and sometimes multiset variables, which will be clear from the text. Small roman letters $p, q, \ldots$ possibly with indices are reserved for atomic formulas (atoms) in the language. For a formula A, the formula $\neg A$ is defined as $A \to \bot$ and the formula $\bigcirc^n A$ is recursively defined by $\bigcirc^0 A = A$ and $\bigcirc^{n+1} A = \bigcirc \bigcirc^n A$. And for a multiset Γ, by $\bigcirc \Gamma$ we mean $\{\bigcirc \gamma \mid \gamma \in \Gamma\}$, where $\bigcirc \in \{\Box, \Diamond\}$. By a *rule* we mean an expression of the form:

$$\frac{S_1 \quad \ldots \quad S_n}{S}$$

where S, called the *conclusion*, and S_i's, called the *premises*, are sequents in the form

$$\Gamma_1, \cdots, \Gamma_m, A_1, \cdots, A_n \Rightarrow B_1, B_2, \cdots, B_r, \Delta_1, \cdots, \Delta_s,$$

where Γ_i's and Δ_j's are variables for multisets and A_a's and B_b's are formulas. By an *instance* of a rule or an initial sequent we mean a substitution for all multiset variables and atomic formulas. By a *(sequent) calculus* $\mathbf{G}$ we mean a set of rules. We denote the corresponding logic of the sequent calculus $\mathbf{G}$ by G. For a set of rules $\mathcal{R}$, by $\mathbf{G}+\mathcal{R}$ we simply mean the calculus obtained by adding every rule in $\mathcal{R}$ to $\mathbf{G}$. A *proof* π in the calculus $\mathbf{G}$ for a sequnet S is defined as a sequence of sequents $\{S_i\}_{i=1}^m$ such that $S_m = S$ and each sequent S_i is an instance of an initial sequent in $\mathbf{G}$ or derived from an instance of a rule in the system $\mathbf{G}$ from some $S_{j_1}, \ldots, S_{j_k}$ where $j_1, \ldots, j_k < i$. If π is a proof of the sequent S in $\mathbf{G}$ we write $\mathbf{G} \vdash^\pi S$, and sometimes we call π a $\mathbf{G}$-proof of S. The proof that we defined is sometimes also called a *general* or *dag-like* proof. A proof is called *tree-like* if every sequent in the proof is used at most once as a hypothesis of a rule in the proof. *Length* (or *size*) of a formula A or a proof π is defined as the number of symbols in it, and denoted by $|A|$ and $|\pi|$, respectively. Each S_i is called a *line* in the proof. By an *extension* of a calculus $\mathbf{G}$, we mean a calculus $\mathbf{H}$ in which all the initial sequents and rules of G are derivable feasibly. More precisely, there exists a polynomial time algorithm $\mathcal{A}$ that for any rule in $\mathbf{G}$, reads the premises as input and outputs a proof of the conclusion. Moreover, the length of this proof is also polynomial in the length of the input. The algorithm $\mathcal{A}$ also provides a proof for each initial sequent of $\mathbf{G}$, polynomial in the length of the sequent. In this paper, we use feasible and polynomial time interchangeably. From the complexity theoretic point of view, usually it makes a difference if we use the tree-like proofs or the dag-like proofs. However, in the presence of the cut rule, conjunction and implication with their intuitionistic rules, it is possible to simulate the DAG-like proofs by the tree-like ones, feasibly []. Therefore, throughout this paper, we always use the tree-like version of the proof systems for simplicity.

For the intutionistic propositional logics, IPC, we use the multi-conclusion Gentzen-style sequent calculus, $\mathbf{LJ}$, which has the following rules and initial sequents:

Initial sequents:

$$\Gamma, A \Rightarrow A, \Delta \quad , \quad \Gamma, \bot \Rightarrow \Delta \quad , \quad \Gamma \Rightarrow \top, \Delta$$

Structural Rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} (Lw) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} (Rw)$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} (Lc) \quad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta} (Rc)$$

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} (cut)$$

Propositional Rules:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} (L\wedge_1) \quad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} (L\wedge_2) \quad \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} (R\wedge)$$

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} (L\vee) \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} (R\vee_1) \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} (R\vee_2)$$

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta} (L \to) \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} (R \to)$$

The system **LK** is defined as **LJ**, replacing the rule $(R \to)$ with:

$$\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta} (R \to)_c$$

The most basic intuitionistic modal calculus that we are interested in is the calculus $i\mathbf{K}$ defined as $\mathbf{LJ} + \{K_\Box, K_\Diamond\}$:

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} K_\Box \quad \frac{\Gamma, A \Rightarrow B}{\Box\Gamma, \Diamond A \Rightarrow \Diamond B} K_\Diamond$$

Note that the necessitation rule (from the premise $\Rightarrow A$ infer $\Rightarrow \Box A$) is an instance of the rule $(K_\Box)$, and hence present in $i\mathbf{K}$.

To define some basic intuitionistic modal logics, consider the following set of intuitionistic modal rules:

$$\frac{\Gamma \Rightarrow \Diamond\bot, \Delta}{\Gamma \Rightarrow \bot, \Delta} \Diamond\bot \quad \frac{\Gamma \Rightarrow \Diamond(A \vee B), \Delta}{\Gamma \Rightarrow \Diamond A, \Diamond B, \Delta} \Diamond\vee \quad \frac{\Gamma, \Diamond A \Rightarrow \Box B}{\Gamma \Rightarrow \Box(A \to B)} \Box\to$$

$$\frac{\Gamma \Rightarrow \Box A, \Delta}{\Gamma \Rightarrow A, \Delta} T_a \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta} T_b \quad \frac{\Gamma \Rightarrow \Diamond\Box A, \Delta}{\Gamma \Rightarrow A, \Delta} B_a \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Box\Diamond A, \Delta} B_b$$

$$\frac{\Gamma \Rightarrow \Box A, \Delta}{\Gamma \Rightarrow \Box\Box A, \Delta}\, 4_a \qquad \frac{\Gamma \Rightarrow \Diamond\Diamond A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta}\, 4_b \qquad \frac{\Gamma \Rightarrow \Diamond\Box A, \Delta}{\Gamma \Rightarrow \Box A, \Delta}\, 5_a \qquad \frac{\Gamma \Rightarrow \Diamond A, \Delta}{\Gamma \Rightarrow \Box\Diamond A, \Delta}\, 5_a$$

$$\frac{}{\Gamma, \Box\bot \Rightarrow \Delta}\, D_a \qquad \frac{}{\Gamma \Rightarrow \Diamond\top, \Delta}\, D_b \qquad \frac{\Gamma \Rightarrow \Box p, \Delta}{\Gamma \Rightarrow \Diamond p, \Delta}\, D$$

For $1 \leqslant m < n$

$$\frac{\Gamma \Rightarrow \Box^n p, \Delta}{\Gamma \Rightarrow \Box^m p, \Delta}\, 4_{n,m,a} \qquad \frac{\Gamma \Rightarrow \Diamond^m p, \Delta}{\Gamma \Rightarrow \Diamond^n p, \Delta}\, 4_{n,m,b}$$

$$\frac{\langle \Gamma \Rightarrow \Box^i p, \Delta \rangle_{i=0}^n}{\Gamma \Rightarrow \Box^{n+1} p, \Delta}\, tra_{n,a} \qquad \frac{\Gamma \Rightarrow \Diamond^{n+1} p, \Delta}{\Gamma \Rightarrow \{\Diamond^i p\}_{0 \leqslant i \leqslant n}, \Delta}\, tra_{n,a}$$

$$\frac{\Gamma \Rightarrow \Diamond\Box p, \Delta}{\Gamma \Rightarrow \Box\Diamond p, \Delta}\, ga \qquad \frac{\Gamma \Rightarrow \Diamond^k \Box^l p, \Delta}{\Gamma \Rightarrow \Box^m \Diamond^n p, \Delta}\, ga_{klmn}$$

$$\frac{\langle \Gamma \Rightarrow \Diamond p_i, \Delta \rangle_{i=0}^n}{\Gamma \Rightarrow \{\Diamond(p_i \wedge (p_j \vee \Diamond p_j))\}_{i \neq j}, \Delta}\, BW_{n,a} \qquad \frac{\langle \Gamma \Rightarrow \Box(p_i \vee (p_j \wedge \Box p_j)), \Delta \rangle_{i \neq j}}{\Gamma \Rightarrow \{\Diamond p_i\}_{i=0}^n, \Delta}\, BW_{n,b}$$

Define $bd_1^a = \Diamond\Box p$ and $bd_{n+1}^a = \Diamond(\Box p_{n+1} \wedge bd_n^a \wedge \neg p_n)$ and consider the rules:

$$\frac{\Gamma \Rightarrow bd_n^a, \Delta}{\Gamma \Rightarrow p_{n+1}, \Delta}\, BD_{n,a}$$

$$\frac{\Gamma \Rightarrow p, \Delta}{\Gamma \Rightarrow \Box(\Diamond p \to p), \Delta}\, H_a$$

$$\frac{\Gamma \Rightarrow \Diamond(\Box p \wedge q), \Delta}{\Gamma \Rightarrow \Box(\Diamond p \vee q), \Delta}\, dir$$

The calculus **BLL** is defined as **LJ** plus the following rules

$$\frac{\Gamma, A \Rightarrow B}{\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Diamond A}$$

If we add the rule (4_b) to the calculus **BLL**, the corresponding logic of this sequent calculus is called the *propositional lax logic*. As mentioned in the introduction, the modalities $\Box$ and $\Diamond$ are not supposed to be dual of each other in intuitionistic modal logics. Based on this fact, there are many possibilities to define intuitionistic modal logics. In the following, we will introduce some well-known basic intuitionistic modal logics (e.g., see [19]) using the rules defined in Table **??**. Take the language $\mathcal{L}_\Box$ and define the sequent calculus $\mathbf{IntK}_\Box$ as **LJ** plus the rule $(K_\Box)$. The corresponding logic is denoted by

$\mathsf{IntK}_\Box$. Then, we can define $\Diamond\phi \equiv \neg\Box\neg\phi$. As a consequence, $\Diamond$ is not distributed over disjunction and this relation between $\Box$ and $\Diamond$ is too binding from the intuitionistic perspective. Another interesting logic is defined by Fischer Servi ([10], [11]). She considered the language $\mathcal{L}_{\Box\Diamond}$, where $\Box$ and $\Diamond$ are distinct modalities, and imposed mild connections between $\Box$ and $\Diamond$ and defined the logic FS with the following sequent calculus:

$$\mathbf{FS} := \mathbf{LJ} + \{(K_\Box), (K_\Diamond), (\Diamond\bot), (\Diamond\vee), (\Box\to)\}.$$

She claimed that the logic FS is the true intuitionistic analogue of the classical modal logic K. She provided two evidences to support her claim. First, she mapped FS to an extension of the fusion of K and S4, using a natural generalization of Gödel's translation. Second, using the standard translation of modal formulas to first order formulas, in the same manner that K is mapped into the classical first order logic, FS is mapped into the intuitionistic first order logic. Several extensions of FS have been introduced and studied. One important one is the logic MIPC whose sequent calculus is defined as follows:

$$\mathbf{MIPC} := \mathbf{FS} + \{(T_a), (T_b), (4_a), (4_b), (5_a), (5_b)\}.$$

Two other interesting logics are the intuitionistic versions of the classical modal logics S4 and S5, denoted by IS4 and IS5, respectively [17].

Theorem 2.1. *If* $\mathbf{G}$ *extends* $\mathbf{IPC}$*, then all rules of* $\mathbf{LJ}$ *are feasibly admissible in* $\mathbf{G}$*, i.e., for any rule in* $\mathbf{LJ}$

$$\frac{S_1 \cdots S_n}{S}$$

there exists a polynomial time algorithm f *such that for any* $\mathbf{G}$*-proofs* π_i *for* S_i*,* $f(\pi_1, \cdots, \pi_n)$ *is a* $\mathbf{G}$*-proof of* S*.*

Proof. It is easy! $\Box$

3 Almost Positive Rules

In this section, we introduce almost positive rules. These rules have a general form and for our main result we only consider sequent calculi consisting of these rules. First, we need to define formulas of a special form.

Definition 3.1. The following sets of formulas are defined in $\mathcal{L}$.

- The set of *basic* formulas is the smallest set containing atomic formulas, the constants $\top$ and $\bot$ and closed under $\{\wedge, \vee, \Diamond\}$.

- The set of *almost positive (a.p.)* formulas is the smallest set containing basic formulas and closed under $\{\wedge, \vee, \Box, \Diamond\}$ and implications of the form $A \to B$, where A is basic and B is almost positive.

- The set of *almost negative (a.n.)* formulas is the smallest set containing basic formulas and closed under $\{\wedge, \Box\}$ and implications of the form $A \to B$, where A is almost positive and B is almost negative.

A formula in the languages $\mathcal{L}_\Box$, $\mathcal{L}_\Diamond$ and $\mathcal{L}_p$ is called basic, almost positive or almost negative, if it is basic, almost positive or almost negative as a formula in the bigger language $\mathcal{L}$.

Example 3.2. In the following, we will discuss various examples.

- Examples of formulas that are basic and hence both almost positive and almost negative are: $p \wedge q$, $p \vee q$, and $\Diamond p$.

- Examples of formulas that are *not* basic but they are both almost positive and almost negative are: $\Box p$, $p \to q$, and $\neg p$. In fact, it is easy to see by the definition that for basic formulas A and B, both $A \to B$ and $\Box A$ are both almost positive and almost negative.

- Some examples of formulas that are almost positive but they are not almost negative are: $\Diamond\Box p, \Box p \vee q, q \to \Diamond\Box p$, and $p \vee \neg p$.

- And some examples of formulas that are almost negative but they are not almost positive are: $\Box p \to q, \Diamond\Box p \to q$, and $\neg\neg p$.

In the next definition, we tend the letters M and N to be reminiscent of the formulas almost positive and almost negative, respectively.

Definition 3.3. Let $\overline{M}, \overline{M'_i}, \overline{N}, \overline{N'_i}$ be multisets of formulas, where $\overline{M}$ and $\overline{M'_i}$ only consist of almost positive formulas and $\overline{N}$ and $\overline{N'_i}$ only consist of almost negative formulas, for $1 \leqslant i \leqslant n$. A rule is called

- *left almost positive*, when it is of the form

$$\frac{\{\Gamma, \overline{N'_i} \Rightarrow \overline{M'_i}, \Delta\}_{i=1}^{n}}{\Gamma, \overline{M} \Rightarrow \Delta}$$

  with the condition that if $n > 1$, then all formulas in $\overline{N'_i}$ are basic (it means that only when $n = 1$, the formulas in $\overline{N'_1}$ can be almost negative formulas that are not basic), and

- *right almost positive*, when it has one of the following forms

$$\frac{\{\Gamma, \overline{N'_i} \Rightarrow \overline{M'_i}\}_{i=1}^n}{\Gamma \Rightarrow \overline{N}} \text{ (context-free)} \qquad \frac{\{\Gamma \Rightarrow \overline{M'_i}, \Delta\}_{i=1}^n}{\Gamma \Rightarrow \overline{N}, \Delta} \text{ (contextual)}$$

with the condition that $\overline{N'_i}$ consist of basic formulas. Moreover, if $\overline{N}$ has at most one formula, then it can be almost negative. Otherwise, if $\overline{N}$ has more than one formula, then all of them must be basic.
A rule is called almost positive over the languages $\mathcal{L}_\Box$, $\mathcal{L}_\Diamond$ or $\mathcal{L}_p$ if it is an almost positive rule over the bigger language $\mathcal{L}$.

Example 3.4. All the rules in Table **??** are almost positive. The following are examples of rules that are not almost positive.

$$\frac{}{\Gamma \Rightarrow p \vee \neg p} \qquad \frac{}{\Gamma \Rightarrow p, \neg p} \qquad \frac{\Gamma \Rightarrow \neg\neg p}{\Gamma \Rightarrow p}$$

$$\frac{\Gamma, \neg p \Rightarrow \bot}{\Gamma \Rightarrow p} \qquad \frac{\Gamma, p \Rightarrow \Delta \qquad \Gamma, \neg p \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Let us explain the last rule (i.e., the right rule in the second row). Based on the form of the rules in Definition 3.1, if it is an almost positive rule, it must be of the form of a left rule. However, since there are two premises in the rule, both formulas in the antecedents must be basic, which is not the case here, since $\neg p$ is not basic. The other rules are examples of right rules that do not satisfy the conditions mentioned in Definition 3.1. The reason, which can be easily checked, is that $p \vee \neg p$ is not almost negative, $\neg p$ is not basic, and $\neg\neg p$ is not almost positive. These rules show how not satisfying the conditions in Definition 3.1 will result in emerging some rules that are not acceptable intuitionistically.

4 Feasible Visser-Harrop Property

This section contains the main results of the paper. The goal of this section is to show that any sequent calculus consisting of almost positive rules and cut has feasible Visser-Harrop property. To achieve this goal, first we will enlarge the language $\mathcal{L}$ with some fresh variables and define a translation function from $\mathcal{L}$ to this new language. Theorem 4.5 shows how the translation function commutes with basic, a.p., and a.n. formulas. Then, in the main theorem, Theorem 4.12, we consider a sequent calculus only consisting of almost positive rules and extending $i\mathbf{K}$. Then, we show that if a sequent is provable in this sequent calculus, then the translation of the sequent (with the aid of a *harmless* set of formulas) is also provable in it.

Definition 4.1. For any formula ϕ in $\mathcal{L}$, define $\langle \phi \rangle$ as a new propositional variable, which we call an *angled atom*, and add it to the language $\mathcal{L}$. We denote the new language with all the angled atoms by $\mathcal{L}^+$.

Definition 4.2. The translation function $t : \mathcal{L} \to \mathcal{L}^+$ is defined in the following way:

- $\bot^t = \bot$, $\top^t = \langle \top \rangle$, and for any atomic formula p define $p^t = \langle p \rangle$;
- $(A \circ B)^t = (A^t \circ B^t) \wedge \langle A \circ B \rangle$, where $\circ \in \{\wedge, \vee, \to\}$;
- $(\bigcirc A)^t = (\bigcirc A^t) \wedge \langle \bigcirc A \rangle$, where $\bigcirc \in \{\Box, \Diamond\}$.

For a multiset Γ, by Γ^t we mean the multiset consisting of the translation of all the elements of Γ, i.e., $\Gamma^t = \{\gamma^t \mid \gamma \in \Gamma\}$. Note the difference between the translation of $\bot$ and $\top$, which will be more clear in the proof of Theorem 4.5. The substitution $s : \mathcal{L}^+ \to \mathcal{L}$ is called *standard* when it maps the angled atom $\langle \phi \rangle$ to ϕ and preserves the non-angled atomic formulas. It is easy to check that s can be seen as a translation function from $\mathcal{L}^+$ to $\mathcal{L}$ canceling all the changes made by the translation t, and tracing back the original formula. For instance, for any formula $A \in \mathcal{L}$, we have $A(\langle \phi_1 \rangle, \cdots, \langle \phi_n \rangle)^s = A(\phi_1, \cdots, \phi_n)$, using induction on the structure of A, and by definition. Finally, Γ^s is defined in the usual way, namely $\{\gamma^s \mid \gamma \in \Gamma\}$.

Lemma 4.3. *For any formula A in the language $\mathcal{L}$, there exists a proof π such that $i\mathbf{K} \vdash^{\pi} A^t \Rightarrow \langle A \rangle$. Moreover, the process of finding π from A is polynomial time computable.*

Proof. We first propose an algorithm $\mathcal{A}$ to find the proof π and then we will address its feasibility. The algorithm works in the following manner: It reads A as the input and outputs π. If A is atomic (including $\bot$ and $\top$), then it writes $A^t \Rightarrow A$ as the proof. Using Definition 4.2, A^t is either $\langle A \rangle$ or $\bot$. In both cases, the sequent $A^t \Rightarrow A$ is an instance of an initial sequent in $i\mathbf{K}$ and hence provable. The length of this sequent is constant (at most 7), because the length of A is one. Otherwise, if A is not atomic, then A is either of the form $B \circ C$ or $\bigcirc B$, where $\circ \in \{\wedge, \vee, \to\}$ and $\bigcirc \in \{\Box, \Diamond\}$. Using Definition 4.2 we have $(B \circ C)^t = (B^t \circ C^t) \wedge \langle B \circ C \rangle = (B^t \circ C^t) \wedge \langle A \rangle$ and $(\bigcirc B)^t = \bigcirc B^t \wedge \langle \bigcirc B \rangle = \bigcirc B^t \wedge \langle A \rangle$. In any of these cases, using the rule $(L\wedge_2)$ the algorithm outputs the following proof π

$$\frac{\langle A \rangle \Rightarrow \langle A \rangle}{D \wedge \langle A \rangle \Rightarrow \langle A \rangle}\,(L\wedge_2)$$

where D is either $B^t \circ C^t$ or $\bigcirc B^t$, depending on the case. Therefore, we have shown that $i\mathbf{K} \vdash A^t \Rightarrow \langle A \rangle$. Now, let us analyze the complexity of the algorithm. If we calculate the complexity of A^t, then we can easily find the complexity of the algorithm. Using recursion, when we unfold all the translated formulas in the definition of A^t, in the end we reach a formula that only consists of angled atoms $\langle B \rangle$, where B is a subformula of A, connectives and modalities. It is easy to check that the number of these angles atoms is less than or equal to the number of subformulas of A, which is at most $|A|$. The length of each angled atom is also at most $|A| + 2$. The number of connectives and modalities are also each at most $|A|$. Therefore, we have

$$|A^t| \leqslant |A|(|A| + 2) + |A| + |A| = |A|^2 + 4|A|.$$

Hence, the length of the proof π is at most $|A|^2+8|A|+11$, which is polynomial in $|A|$. This concludes the proof of the feasibility of the algorithm. $\square$

When we talk about an atom p in the language $\mathcal{L}^+$, we mean either the atomic formulas in $\mathcal{L}$ or the new angled atoms.

Definition 4.4. The set of *implicational Horn formulas* is the smallest set of $\mathcal{L}^+$-formulas containing atomic formulas in $\mathcal{L}^+$ and is closed under implications of the form $\bigwedge Q \to p$, where $Q = \{q_1, \ldots, q_n\}$ is a multiset of atoms and p an atom in $\mathcal{L}^+$ and by $\bigwedge Q$ we mean $q_1 \wedge \ldots \wedge q_n$. The set of *modal Horn* formulas is the smallest set of formulas containing atomic formulas in $\mathcal{L}^+$ which is closed under $\Box$ and closed under implications of the form $A \to B$ where A is of the form $\bigwedge_{i=1}^{k} \Diamond^{n_i} p$ and B is a modal Horn formula.

As mentioned earlier, by $\Diamond^0 p$ we mean p. Therefore, it is easy to see that any implicational Horn formula is also a modal Horn formula.

The following theorem is one of the main tools in proving the feasible Visser-Harrop property. It expresses how the translation t commutes with the formulas based on their form. In each case, whether the formula is basic, a.p. or a.n., a set of modal Horn formulas (depending on the formula itself) is needed as an assistance to make the sequent provable in $i\mathbf{K}$. For basic formulas, both directions are provable. However, for a.p. and a.n. formulas, only one direction can be proved, and in the case of a.n. formulas, even another additional formula is needed to make the sequent provable.

Theorem 4.5. (*i*) *For any basic formula $A(\overline{p}) \in \mathcal{L}$ and any formulas $\overline{\phi} \in \mathcal{L}^+$, there is a set of modal Horn formulas $\Phi_{A,\overline{\phi}}$ constructed from angled atoms such that the sequents*

$$\Phi_{A,\overline{\phi}}, (A(\overline{\phi}))^t \Rightarrow A(\overline{\phi^t}) \quad \textit{and} \quad \Phi_{A,\overline{\phi}}, A(\overline{\phi^t}) \Rightarrow (A(\overline{\phi}))^t$$

are feasibly provable in $i\mathbf{K}$ *in the lengths of* A *and* $\overline{\phi}$.

(*ii*) *For any almost positive formula* $A(\overline{p})$ *and any formulas* $\overline{\phi} \in \mathcal{L}^+$, *there is a set of modal Horn formulas* $\Pi_{A,\overline{\phi}}$ *constructed from angled atoms such that*

$$\Pi_{A,\overline{\phi}}, (A(\overline{\phi}))^t \Rightarrow A(\overline{\phi^t})$$

is feasibly provable in $i\mathbf{K}$ *in the lengths of* A *and* $\overline{\phi}$.

(*iii*) *For any almost negative formula* $A(\overline{p})$ *and any formulas* $\overline{\phi} \in \mathcal{L}^+$, *there is a set of modal Horn formulas* $\Upsilon_{A,\overline{\phi}}$ *constructed from angled atoms such that*

$$\Upsilon_{A,\overline{\phi}}, \langle A(\overline{\phi})\rangle, A(\overline{\phi^t}) \Rightarrow (A(\overline{\phi}))^t$$

is feasibly provable in $i\mathbf{K}$ *in the lengths of* A *and* $\overline{\phi}$.

In each case, there exists a proof π_A *such that* $i\mathbf{K} \vdash^{\pi_A} \Rightarrow \bigwedge \Theta^s_{A,\overline{\phi}}$, *where* s *is the standard substitution. Moreover, the processes of finding* $\Theta_{A,\overline{\phi}}$ *and* π_A *are polynomial time computable in the lengths of* A *and* $\overline{\phi}$, *for* $\Theta \in \{\Phi, \Pi, \Upsilon\}$.

Proof. We may use A for $A(\overline{\phi})$, B for $B(\overline{\phi})$, and C for $C(\overline{\phi})$, when no confusion occurs. We will treat each case separately using induction on the structure of the formula $A(\overline{p})$. We start with (*i*) where $A(\overline{p})$ is a basic formula.

(*i*) : For the case that $A(\overline{p})$ is either an atom p or $\bot$, for any formulas $\overline{\phi}$ we take the set $\Phi_{A,\overline{\phi}}$ to be the empty set. Then the sequents are trivially true in $i\mathbf{K}$. Moreover, since $\bigwedge \varnothing$ is defined as $\top$, we have $i\mathbf{K} \vdash \Rightarrow \bigwedge \Phi^s_{A,\overline{\phi}}$. For the case that A is $\top$ then we take $\Phi_{A,\overline{\phi}}$ to be $\{\langle\top\rangle\}$, which is by definition a set of modal Horn formulas. The sequents in this case will become $\langle\top\rangle, \langle\top\rangle \Rightarrow \top$ and $\langle\top\rangle, \top \Rightarrow \langle\top\rangle$, which are both provable in $i\mathbf{K}$. Moreover, $\bigwedge \Phi^s_{A,\overline{\phi}}$ is equal to $\top$ and hence $i\mathbf{K} \vdash \Rightarrow \bigwedge \Phi^s_{A,\overline{\phi}}$. Now, suppose $A(\overline{p}) = B(\overline{p}) \wedge C(\overline{p})$. By Definition 4.2, we have $(A(\overline{\phi}))^t = (B(\overline{\phi}))^t \wedge (C(\overline{\phi}))^t \wedge \langle B(\overline{\phi}) \wedge C(\overline{\phi})\rangle$. Using induction hypothesis, there exist multisets $\Phi_{B,\overline{\phi}}$ and $\Phi_{C,\overline{\phi}}$ such that

$$\Phi_{B,\overline{\phi}}, (B(\overline{\phi}))^t \Rightarrow B(\overline{\phi^t}) \quad (1) \quad , \quad \Phi_{B,\overline{\phi}}, B(\overline{\phi^t}) \Rightarrow (B(\overline{\phi}))^t \quad (2),$$

$$\Phi_{C,\overline{\phi}}, (C(\overline{\phi}))^t \Rightarrow C(\overline{\phi^t}) \quad (3) \quad , \quad \Phi_{C,\overline{\phi}}, C(\overline{\phi^t}) \Rightarrow (C(\overline{\phi}))^t \quad (4).$$

are provable in $i\mathbf{K}$. Let us first investigate the trickier sequent, namely $\Phi_{A,\overline{\phi}}, A(\overline{\phi^t}) \Rightarrow (A(\overline{\phi}))^t$. Using the rules in $i\mathbf{K}$, we easily get from (2) and (4)

$$i\mathbf{K} \vdash \Phi_{B,\overline{\phi}}, \Phi_{C,\overline{\phi}}, B(\overline{\phi^t}) \wedge C(\overline{\phi^t}) \Rightarrow (B(\overline{\phi}))^t \wedge (C(\overline{\phi}))^t \quad (5).$$

By Lemma 4.3, both $(B(\overline{\phi}))^t \Rightarrow \langle B(\overline{\phi})\rangle$ and $(C(\overline{\phi}))^t \Rightarrow \langle C(\overline{\phi})\rangle$ are feasibly provable in $i\mathbf{K}$, hence is also $(B(\overline{\phi}))^t \wedge (C(\overline{\phi}))^t \Rightarrow \langle B(\overline{\phi})\rangle \wedge \langle C(\overline{\phi})\rangle$ (6). We claim that taking $\Phi_{A,\overline{\phi}}$ as

$$\Phi_{B,\overline{\phi}} \cup \Phi_{C,\overline{\phi}} \cup \{\langle B(\overline{\phi})\rangle \wedge \langle C(\overline{\phi})\rangle \to \langle B(\overline{\phi}) \wedge C(\overline{\phi})\rangle\}$$

works. Note that $\Phi_{A,\overline{\phi}}$ only consists of modal Horn formulas constructed from angled atoms. Using

$$\langle B(\overline{\phi})\rangle \wedge \langle C(\overline{\phi})\rangle \to \langle B(\overline{\phi}) \wedge C(\overline{\phi})\rangle, \langle B(\overline{\phi})\rangle \wedge \langle C(\overline{\phi})\rangle \Rightarrow \langle B(\overline{\phi}) \wedge C(\overline{\phi})\rangle,$$

and (5) and (6) we finally get

$$i\mathbf{K} \vdash \Phi_{A,\overline{\phi}}, A(\overline{\phi^t}) \Rightarrow (A(\overline{\phi}))^t.$$

The other sequent, $\Phi_{A,\overline{\phi}}, (A(\overline{\phi}))^t \Rightarrow A(\overline{\phi^t})$, is easier. Using the rules in $i\mathbf{K}$, we easily get from (1) and (3)

$$i\mathbf{K} \vdash \Phi_{B,\overline{\phi}}, \Phi_{C,\overline{\phi}}, (B(\overline{\phi}))^t \wedge (C(\overline{\phi}))^t \Rightarrow B(\overline{\phi^t}) \wedge C(\overline{\phi^t}).$$

Then, using the rule (Lw) we can introduce $\langle B(\overline{\phi}) \wedge C(\overline{\phi})\rangle$ and $\langle B(\overline{\phi})\rangle \wedge \langle C(\overline{\phi})\rangle \to \langle B(\overline{\phi}) \wedge C(\overline{\phi})\rangle$ in the antecedent and we obtain

$$i\mathbf{K} \vdash \Phi_{A,\overline{\phi}}, (A(\overline{\phi}))^t \Rightarrow A(\overline{\phi^t}).$$

By $i\mathbf{K} \vdash \Rightarrow (\langle B(\overline{\phi})\rangle \wedge \langle C(\overline{\phi})\rangle \to \langle B(\overline{\phi}) \wedge C(\overline{\phi})\rangle)^s$ and induction hypothesis, we have $i\mathbf{K} \vdash \Rightarrow \bigwedge \Phi^s_{A,\overline{\phi}}$.

In a similar way, we can prove that in the case that $A(\overline{p}) = B(\overline{p}) \vee C(\overline{p})$ taking $\Phi_{A,\overline{\phi}}$ as

$$\Phi_{B,\overline{\phi}} \cup \Phi_{C,\overline{\phi}} \cup \{\langle B(\overline{\phi})\rangle \to \langle B(\overline{\phi}) \vee C(\overline{\phi})\rangle, \langle C(\overline{\phi})\rangle \to \langle B(\overline{\phi}) \vee C(\overline{\phi})\rangle\}$$

works. Now, let us consider the final case where $A(\overline{p}) = \Diamond B(\overline{p})$. By Definition 4.3 we have $(A(\overline{\phi}))^t = \Diamond(B(\overline{\phi}))^t \wedge \langle \Diamond B(\overline{\phi})\rangle$. By induction hypothesis, there exists a multiset $\Phi_{B,\overline{\phi}}$ such that

$$\Phi_{B,\overline{\phi}}, (B(\overline{\phi}))^t \Rightarrow B(\overline{\phi^t}) \quad (7) \quad , \quad \Phi_{B,\overline{\phi}}, B(\overline{\phi^t}) \Rightarrow (B(\overline{\phi}))^t \quad (8).$$

hold in $i\mathbf{K}$. Let us investigate the more complicated case, namely the provability of the sequent $\Phi_{A,\overline{\phi}}, A(\overline{\phi^t}) \Rightarrow (A(\overline{\phi}))^t$ in $i\mathbf{K}$. The other sequent can be proved easier. We claim that setting

$$\Phi_{A,\overline{\phi}} = \Box \Phi_{B,\overline{\phi}} \cup \{\Diamond \langle B(\overline{\phi})\rangle \to \langle \Diamond B(\phi)\rangle\}$$

works. First note that $\Phi_{A,\overline{\phi}}$ consists of modal Horn formulas constructed from angled atoms. Moreover, from $i\mathbf{K} \vdash \Rightarrow \bigwedge(\Phi_{B,\overline{\phi}})^s$, the necessitation rule, and $i\mathbf{K} \vdash \Rightarrow (\Diamond\langle B(\overline{\phi})\rangle \to \langle \Diamond B(\phi)\rangle)^s$, we get $i\mathbf{K} \vdash \Rightarrow \bigwedge(\Phi_{A,\overline{\phi}})^s$. Now, to ensure that this choice of $\Phi_{A,\overline{\phi}}$ works, we proceed as follows. Using the rule $(K_\Diamond)$ on (8) we get

$$\Box\Phi_{B,\overline{\phi}}, \Diamond B(\overline{\phi^t}) \Rightarrow \Diamond(B(\overline{\phi}))^t \quad (9).$$

By Lemma 4.3 we have $(B(\overline{\phi}))^t \Rightarrow \langle B(\overline{\phi})\rangle$. Using the rule $(K_\Diamond)$ we get $\Diamond(B(\overline{\phi}))^t \Rightarrow \Diamond\langle B(\overline{\phi})\rangle$. Therefore, using cut and (9) we get $\Box\Phi_{B,\overline{\phi}}, \Diamond B(\overline{\phi^t}) \Rightarrow \Diamond\langle B(\overline{\phi})\rangle$. Again, by cut on the provable sequent $\Diamond\langle B(\overline{\phi})\rangle, \Diamond\langle B(\overline{\phi})\rangle \to \langle \Diamond B(\overline{\phi})\rangle \Rightarrow \langle \Diamond B(\overline{\phi})\rangle$, we get

$$\Box\Phi_{B,\overline{\phi}}, \Diamond B(\overline{\phi^t}), \Diamond\langle B(\overline{\phi})\rangle \to \langle \Diamond B(\overline{\phi})\rangle \Rightarrow \langle \Diamond B(\overline{\phi})\rangle \quad (10).$$

Using the rule (Lw) on (9) and then applying the rule $(R\wedge)$ on the resulted sequent and (10) we get $\Phi_{A,\overline{\phi}}, A(\overline{\phi^t}) \Rightarrow (A(\overline{\phi}))^t$ in $i\mathbf{K}$. This concludes the proof for (i).

(ii) : Here the formula $A(\overline{p})$ is almost positive. Again we use induction on the structure of the formula. The base case, where $A(\overline{p})$ is a basic formula, is covered in (i). The cases $A = B \circ C$ or $A = \bigcirc B$ where $\circ \in \{\wedge, \vee\}$ and $\bigcirc \in \{\Box, \Diamond\}$ are very easy and similar to the cases in (i). It is easy to see that in the former cases setting $\Pi_{A,\overline{\phi}} = \Pi_{B,\overline{\phi}} \cup \Pi_{C,\overline{\phi}}$ and in the latter cases $\Pi_{A,\overline{\phi}} = \Box\Pi_{B,\overline{\phi}}$ works. The only remaining case, which is also easy, is when $A(\overline{p}) = B(\overline{p}) \to C(\overline{p})$, where $B(\overline{p})$ is a basic formula and $C(\overline{p})$ is almost positive. Using Definition 4.2 we have $(A(\overline{\phi}))^t = (B(\overline{\phi}))^t \to (C(\overline{\phi}))^t \wedge \langle B(\overline{\phi}) \to C(\overline{\phi})\rangle$. By induction hypothesis there exist multisets $\Phi_{B,\overline{\phi}}$ (using (i)) and $\Pi_{C,\overline{\phi}}$ such that

$$\Phi_{B,\overline{\phi}}, B(\overline{\phi^t}) \Rightarrow (B(\overline{\phi}))^t \quad (11) \quad , \quad \Pi_{C,\overline{\phi}}, (C(\overline{\phi}))^t \Rightarrow C(\overline{\phi^t}) \quad (12)$$

hold in $i\mathbf{K}$. Using the rule $(L\to)$ on (11) and (12) and then the rule $(R\to)$ we get

$$\Phi_{B,\overline{\phi}}, \Pi_{C,\overline{\phi}}, (B(\overline{\phi}))^t \to (C(\overline{\phi}))^t \Rightarrow B(\overline{\phi^t}) \to C(\overline{\phi^t}).$$

Now, using the rule $(L\wedge_1)$ to introduce $\langle B(\overline{\phi}) \to C(\overline{\phi})\rangle$ in the antecedent of the sequent, and setting $\Pi_{A,\overline{\phi}} = \Phi_{B,\overline{\phi}} \cup \Pi_{C,\overline{\phi}}$ we get $\Pi_{A,\overline{\phi}}, (A(\overline{\phi}))^t \Rightarrow A(\overline{\phi^t})$, which completes the proof for (ii).

(iii) : In this case the formula $A(\overline{p})$ is almost negative. Using induction on the structure of $A(\overline{p})$, the base case is covered in item (i). For the case that $A(\overline{p}) = B(\overline{p}) \wedge C(\overline{p})$, take $\Upsilon_{A,\overline{\phi}}$ as

$$\Upsilon_{B,\overline{\phi}} \cup \Upsilon_{C,\overline{\phi}} \cup \{\langle B(\overline{\phi})\rangle \wedge \langle C(\overline{\phi})\rangle \to \langle B(\overline{\phi}) \wedge C(\overline{\phi})\rangle,$$

$$\langle B(\overline{\phi})\rangle \wedge \langle C(\overline{\phi})\rangle \to \langle B(\overline{\phi})\rangle, \langle B(\overline{\phi})\rangle \wedge \langle C(\overline{\phi})\rangle \to \langle C(\overline{\phi})\rangle\},$$

and for the case that $A(\overline{p}) = \Box B(\overline{p})$ take

$$\Upsilon_{A,\overline{\phi}} = \Upsilon_{B,\overline{\phi}} \cup \{\langle \Box B(\overline{\phi})\rangle \to \Box\langle B(\overline{\phi})\rangle\}.$$

It is easy to see that in both cases $\Upsilon_{A,\overline{\phi}}$ works and moreover, it is a multiset of modal Horn formulas and $i\mathbf{K} \vdash \Rightarrow \bigwedge \Theta^s_{A,\overline{\phi}}$. Now, the only case left to investigate is $A(\overline{p}) = B(\overline{p}) \to C(\overline{p})$, where $B(\overline{p})$ is almost positive and $C(\overline{p})$ is almost negative. By Definition 4.3 we have $(A(\overline{\phi}))^t = (B(\overline{\phi}))^t \to (C(\overline{\phi}))^t \wedge \langle B(\overline{\phi}) \to C(\overline{\phi})\rangle$. By induction hypothesis, there exist multisets $\Pi_{B,\overline{\phi}}$ and $\Upsilon_{C,\overline{\phi}}$ such that

$$\Pi_{B,\overline{\phi}}, (B(\overline{\phi}))^t \Rightarrow B(\overline{\phi^t}) \quad (13) \quad , \quad \Upsilon_{C,\overline{\phi}}, \langle C(\overline{\phi})\rangle, C(\overline{\phi^t}) \Rightarrow (C(\overline{\phi}))^t \quad (14)$$

hold in $i\mathbf{K}$. We claim taking

$$\Upsilon_{A,\overline{\phi}} = \Pi_{B,\overline{\phi}}, \Upsilon_{C,\overline{\phi}} \cup \{(\langle B(\overline{\phi}) \to C(\overline{\phi})\rangle \wedge \langle B(\overline{\phi})\rangle) \to \langle C(\overline{\phi})\rangle\}$$

works. Applying the rule $(L \to)$ on (13) and (14) we have

$$\Pi_{B,\overline{\phi}}, \Upsilon_{C,\overline{\phi}}, (B(\overline{\phi}))^t, \langle C(\overline{\phi})\rangle, B(\overline{\phi^t}) \to C(\overline{\phi^t}) \Rightarrow (C(\overline{\phi}))^t.$$

Using the cut rule on the above sequent and $(B(\overline{\phi}))^t, (B(\overline{\phi}))^t \to \langle C(\overline{\phi})\rangle \Rightarrow \langle C(\overline{\phi})\rangle$ and the contraction rule and $(R \to)$ we get

$$\Pi_{B,\overline{\phi}}, \Upsilon_{C,\overline{\phi}}, (B(\overline{\phi}))^t \to \langle C(\overline{\phi})\rangle, B(\overline{\phi^t}) \to C(\overline{\phi^t}) \Rightarrow (B(\overline{\phi}))^t \to (C(\overline{\phi}))^t \quad (15)$$

On the other hand, using the cut rule on the provable sequents

$$\langle A\rangle, \langle A\rangle \wedge \langle B\rangle \to \langle C\rangle \Rightarrow \langle B\rangle \to \langle C\rangle \quad , \quad \langle B\rangle \to \langle C\rangle \Rightarrow (B(\overline{\phi}))^t \to \langle C\rangle$$

we get

$$\langle A\rangle, \langle A\rangle \wedge \langle B\rangle \to \langle C\rangle \Rightarrow (B(\overline{\phi}))^t \to \langle C\rangle$$

in $i\mathbf{K}$. Using the cut rule on the above sequent and (15) we obtain

$$\Pi_{B,\overline{\phi}}, \Upsilon_{C,\overline{\phi}}, \langle A\rangle, \langle A\rangle \wedge \langle B\rangle \to \langle C\rangle, B(\overline{\phi^t}) \to C(\overline{\phi^t}) \Rightarrow (B(\overline{\phi}))^t \to (C(\overline{\phi}))^t.$$

Using the left weakening rule on $\langle A\rangle \Rightarrow \langle A\rangle$ and then applying the rule $(R\wedge)$ on the above sequent we get

$$\Upsilon_{A,\overline{\phi}}, \langle A(\overline{\phi})\rangle, A(\overline{\phi^t}) \Rightarrow (A(\overline{\phi}))^t,$$

as required. Again it is clear that $\Upsilon_{A,\overline{\phi}}$ is a multiset of modal Horn formulas and $i\mathbf{K} \vdash \Rightarrow \bigwedge \Theta^s_{A,\overline{\phi}}$. This concludes the proof of part *(iii)*.

The only thing that remains to investigate is the feasibility of the algorithm. Note that in each step of the induction, depending on whether $A = B \circ C$ or $A = \bigcirc B$ for $\circ \in \{\wedge, \vee, \rightarrow\}$ or $\bigcirc \in \{\Box, \Diamond\}$, we have

$$|\Theta_{A,\overline{\phi}}| = |\Theta_{B,\overline{\phi}}| + |\Theta_{C,\overline{\phi}}| + c(|A| + |\overline{\phi}|) \quad \text{or} \quad |\Theta_{A,\overline{\phi}}| = |\Theta_{B,\overline{\phi}}| + c(|A| + |\overline{\phi}|),$$

where $\Theta \in \{\Phi, \Pi, \Upsilon\}$ and c is a constant (take it as the largest number of added symbols appearing in any of the cases). The number of induction steps are at most equal to $|A|$. Therefore, we have $|\Theta_{A,\overline{\phi}}| \leqslant |A| \cdot c(|A| + |\overline{\phi}|)$. Now, let us discuss the length of proofs of the sequents in each case. As an example, we will analyse the proof for the sequent in (iii), the others are similar. Let us denote the proof of the sequent by π_A. The number of lines of π_A, denoted by $\#\pi_A$ is equal to $\#\pi_B + \#\pi_C + d_1$ or $\#\pi_B + d_1$, where d_1 is a constant and is the largest number of the added lines to the proofs of π_B and π_C (or π_B) to get π_A. Using $|\Phi_{A,\overline{\phi}}|$ and Lemma 4.3, it is easy to see that the length of each line of π_A is polynomial in $|A|$ and $|\overline{\phi}|$. Therefore, the processes of finding $\Theta_{A,\overline{\phi}}$ and π_A are polynomial time computable in the lengths of $|A|$ and $|\overline{\phi}|$.

$\square$

Remark 4.6. First, note that Theorem 4.5 holds for *any* multiset of formulas $\overline{\phi}$, as long as the construction of the formula A is as described. Another point to make is that in the proof of Theorem 4.5 other (sometimes simpler) choices exist for the set of modal Horn formulas such that it maked the sequent provable in $i\mathbf{K}$. The importance of our choices for these sets of modal Horn formulas is the condition that the standard translation of each of their elements are provable in $i\mathbf{K}$. We will use this property later in Theorem 4.12.

Lemma 4.7. *Let S be the sequent $\Gamma \Rightarrow \bigvee_{i=1}^{n} p_i$, where Γ is a multiset of implicational Horn formulas and $p_1, p_2, \cdots, p_n$ are $n \geqslant 2$ atomic formulas. If $\mathbf{LK} \vdash \Gamma \Rightarrow \bigvee_{i=1}^{n} p_i$, then, there exist an $1 \leqslant i \leqslant n$ and an $\mathbf{LJ}$-proof π such that $\mathbf{LJ} \vdash^{\pi} \Gamma \Rightarrow p_i$. The processes of finding i and π are polynomial time computable in the length of S.*

Proof. We describe the process of finding i and π and then we will show that it is feasible. The main idea is the unit propagation rule used in the Horn satisfiability algorithm. First, note that Γ only consists of implicational Horn formulas, which means that they are either atomic formulas, which we call units, or formulas of the implicational form $\bigwedge Q \rightarrow a$, where Q is a non-empty sequence of atomic formulas and a an atomic formula. The process of finding i is the following: In the initial step, $k = 0$, set $\Gamma_0 = \Gamma$. In the step $k+1$ produce a multiset Γ_{k+1} in the following way by modifying Γ_k: Pick a new (i.e., not picked in the previous steps) unit r in Γ_k. It is not important which one (for instance, at the beginning the algorithm can set an order on all the atomic formulas in the sequent S, and each time proceeds with respect to this order). Then, check if it is any of p_i's (i.e., any of the atoms in the succedent of the sequent $\Gamma \Rightarrow \bigvee_{i=1}^{n} p_i$). If this is the case, halt and output this p_i. Otherwise, to produce Γ_{k+1}:

- For any $\gamma \in \Gamma$, if it is of the form $\bigwedge Q \to r$, delete γ.
- For any $\gamma \in \Gamma$ of the form $\bigwedge Q \to s$ for $s \neq r$, if r is one of the conjuncts in Q, delete r from the formula. With this process the implication $r \to s$ transforms to s.

Note that in this process the unit r is not deleted and we keep it. We show that the algorithm finally halts, meaning that it finds a p_i as a unit in some stage. For the sake of contradiction, assume the algorithm never halts. First, we show that $\mathbf{LK} \vdash \Gamma_k \Rightarrow \bigvee_{i=1}^{n} p_i$, for each $k \geqslant 0$. We have $\mathbf{LJ} \vdash \bigwedge \Gamma_{k+1} \Leftrightarrow \bigwedge \Gamma_k$, for each k. The reason is that if the chosen unit is r, then $\mathbf{LJ} \vdash r \wedge (\bigwedge Q \to r) \Leftrightarrow r$ and for any $s \neq r$ we have $\mathbf{LJ} \vdash r \wedge (\bigwedge Q \to s) \Leftrightarrow r \wedge (\bigwedge Q' \to s)$, where Q' is Q after deleting r from it. Second, note that in any stage k, there is always a new unit to pick up. Assuming otherwise, it means that before reaching the stage k, we have checked all possible units and as the algorithm has not halted, they did not intersect with $\{p_i\}_{i=1}^{n}$. Moreover, it implies that for any implication in Γ_k in the form $\bigwedge Q \to p$, there is a non-unit in Q, because otherwise if Q consists of units, as each unit has been chosen before, it must have been eliminated from Q before and hence Q must have become empty, which is not the case. Now, to reach a contradiction with $\mathbf{LK} \vdash \Gamma_k \Rightarrow \bigvee_{i=1}^{n} p_i$, define a classical valuation v by setting all the units in Γ_k to one and all the other atoms to zero. It makes all implicational formulas in Γ_k true, because in the antecedent of each of them there should be at least one non-unit atom. It also satisfies all units in Γ_k, by definition. Hence v satisfies Γ_k. However, as no p_i has been occurred as a unit, we have $v(p_i) = 0$. This contradicts $\mathbf{LK} \vdash \Gamma_k \Rightarrow \bigvee_{i=1}^{n} p_i$. Therefore, there is always a new unit and since the number of units are finite, we reach a contradiction with the assumption that the algorithm never halts. Hence, it halts in some stage m and finds some p_i as the unit meaning $p_i \in \Gamma_m$. Hence, $\Gamma_m \vdash p_i$ and since $\mathbf{LJ} \vdash \bigwedge \Gamma_0 \leftrightarrow \bigwedge \Gamma_m$, we obtain $\Gamma = (\Gamma_0 \Rightarrow p_i)$ is provable in $\mathbf{LJ}$.
Now there are two points to make. First, note that the algorithm is polynomial time in the length of $\Gamma \Rightarrow \bigvee_{i=1}^{n} p_i$, just like the original unit propagation algorithm []. The reason simply is that it goes at most for N many steps,

where N is the number of atomic formulas in $\Gamma \Rightarrow \bigvee_{i=1}^{n} p_i$ and in the step k, it just scans Γ_k once and does some small changes that makes it shorter in length. Second, the proof for $\Gamma \Rightarrow p_i$ is constructed by cuts over the **LJ**-proofs $\bigwedge \Gamma_k \Rightarrow \bigwedge \Gamma_{k+1}$ for each $0 \leqslant k < m$ and the fact that $p_i \in \Gamma_m$. Since $m \leqslant N$, the **LJ**-proofs $\bigwedge \Gamma_k \Rightarrow \bigwedge \Gamma_{k+1}$ are easy applications of implication rules. Hence, the intuitionistic proof for $\Gamma \Rightarrow p_i$ is poly-time computable in the length of the sequent $\Gamma \Rightarrow \bigvee_{i=1}^{n} p_i$.
Note that the algorithm uses the classical provability of the sequent $\Gamma \Rightarrow \bigvee_{i=1}^{n} p_i$ and not its proof. Moreover, it is polynomial time in the length of the sequent itself and not its classical proof. □

Definition 4.8. The set of *Harrop* formulas is the smallest set of $\mathcal{L}$-formulas including atomic formulas, $\bot, \top$, and is closed under $\wedge, \Box$, and implications of the form $A \to B$, where A is an arbitrary formula and B is a Harrop formula. A formula in the languages $\mathcal{L}_\Box$, $\mathcal{L}_\Diamond$ and $\mathcal{L}_p$ is called Harrop, if it is Harrop as a formula in the bigger language $\mathcal{L}$.

Lemma 4.9. *If $A \in \mathcal{L}$ is a Harrop formula, then so is $A^t \in \mathcal{L}^+$.*

Proof. The proof is easy by induction on the structure of A. As an example, consider the case where $A = B \to C$, where B is an arbitrary formula and C is a Harrop formula. By induction hypothesis C^t is also a Harrop formula, and hence by definition so is $B^t \to C^t$. Since any atom is Harrop and the set of Harrop formulas is closed under conjunction, we have $(B \to C)^t = (B^t \to C^t) \wedge \langle B \to C \rangle$ is also Harrop. □

Lemma 4.10. *For any Harrop formula $A \in \mathcal{L}$, there exists a multiset Γ_A with the following conditions:*

- *(i) Γ_A consists of modal Horn formulas, constructed only from $\bot, \top$, and angled atoms,*
- *(ii) $i\mathbf{K} \vdash^{\pi_A} \Gamma_A \Rightarrow A^t$, and*
- *(iii) $i\mathbf{K} \vdash^{\pi'_A} \bigwedge \Gamma_A^s \Leftrightarrow A$.*

Furthermore, there exists a polynomial time computable algorithm that reads A and finds Γ_A, π_A and π'_A.

Proof. We first explain the algorithm that constructs Γ_A, π_A and π'_A and then we will check the feasibilty. We define Γ_A by recursion on A and we prove, by using induction on the structure of A, that the required conditions hold for this Γ_A. If A is atomic, or $\bot$, or $\top$, then it is easy to see that $\Gamma = \{A^t\}$ satisfies the conditions and we can define π and π' as the corresponding proofs by

the axioms. If $A = B \wedge C$, where B and C are Harrop formulas, define Γ_A as $\Gamma_B \cup \Gamma_C \cup \{\langle B \wedge C \rangle\}$. By induction hypothesis we have multisets Γ_B and Γ_C, only consisting of modal Horn formulas such that $i\mathbf{K} \vdash^{\pi_B} \Gamma_B \Rightarrow B^t$ and $i\mathbf{K} \vdash^{\pi_C} \Gamma_C \Rightarrow C^t$. It is easy to see that $i\mathbf{K} \vdash^{\pi_A} \Gamma_B, \Gamma_C, \langle B \wedge C \rangle \Rightarrow (B \wedge C)^t$, and π_A and π'_A are the canonical proofs constructed from π_B, π'_B, π_C and π'_C. We can easily check that all the three conditions hold.
If $A = B \to C$, where C is Harrop, we have a multiset for C such that $i\mathbf{K} \vdash \Gamma_C \Rightarrow C^t$, or equivalently $i\mathbf{K} \vdash \bigwedge \Gamma_C \Rightarrow C^t$. By Lemma 4.3, for the formula B, we have $i\mathbf{K} \vdash B^t \Rightarrow \langle B \rangle$. Using the rule $(L \to)$, we get $i\mathbf{K} \vdash B^t, \langle B \rangle \to \bigwedge \Gamma_C \Rightarrow C^t$, and using the rule $(R \to)$, we have $i\mathbf{K} \vdash \langle B \rangle \to \bigwedge \Gamma_C \Rightarrow B^t \to C^t$. Using the left weakening rule and the rule $(R\wedge)$, we get $i\mathbf{K} \vdash \langle B \rangle \to \bigwedge \Gamma_C, \langle B \to C \rangle \Rightarrow (B \to C)^t$. The problem is that the formula $\langle B \rangle \to \bigwedge \Gamma_C$ is not necessarily in the modal Horn form. But fortunately it is possible to write a set of modal Horn formulas equivalent to the conjunction of these two formulas. We claim

$$\Gamma_A = \{\langle B \to C \rangle\} \cup \{\langle B \rangle \to \gamma \mid \gamma \in \Gamma_C\}$$

works. Note that $\langle B \to C \rangle$ is an atom (an angled one) and hence a modal Horn formula. Moreover, any $\gamma \in \Gamma_C$ is either an atom (and in this case $\langle B \rangle \to \gamma$ is implicational Horn), or it is of the form $\bigwedge Q \to p$, where p and each $q \in Q$ are atomic. Then, $\langle B \rangle \to (\bigwedge Q \to p)$ is equivalent to $(\langle B \rangle \wedge \bigwedge Q) \to p$ in $i\mathbf{K}$ and the latter is an implicational Horn formula. For π_A and π'_A, pick the canonical proofs constructed from π_C and π'_C.
If $A = \Box B$, where B is Harrop, define $\Gamma_A = \Box \Gamma_B, \langle \Box B \rangle$. First, obviously Γ_A cosists of modal-Horn formulas, built only from angled atoms. Second, note that using the rule $K_\Box$, left weakening and $(R\wedge)$, we have $i\mathbf{K} \vdash \Box \Gamma_A, \langle \Box B \rangle \Rightarrow (\Box B)^t$. Third, using the fact that $i\mathbf{K} \vdash \bigwedge(\Box \alpha_i) \Leftrightarrow \Box(\bigwedge \alpha_i)$, we can reason $i\mathbf{K} \vdash \bigwedge(\Box \Gamma_B^s) \Leftrightarrow \Box B$, and hence $i\mathbf{K} \vdash \bigwedge \Gamma_A^s \Leftrightarrow \Box B$.
Finally, for the feasibility, first note that by induction on the structure of A, it is easy to prove that the cardinality of Γ_A is linear in the length of $|A|$. Then, using this fact we can use induction again to show that $|\bigwedge \Gamma_A|$ is quadratic in the length of $|A|$ and therefore, both $|\pi_A|$ and $|\pi'_A|$ are also polynomially bounded in $|A|$, because their complexity are controlled by the complexity of Γ_A. Finally, note that the whole algorithm is a recursion on the structure of A and hence it can be seen as a recursion on $|A|$ and in each step everything from constructing Γ_A to defining π_A and π'_A are feasible. Therefore, since Γ_A, π_A and π'_A are polynomially bounded in the length of $|A|$, the whole algorithm is feasible itself. $\Box$

Definition 4.11. We say a sequent calculus $\mathbf{G}$ has *the feasible Visser-Harrop property*, if there exists a poly-time algorithm that reads a proof π of $\Gamma, \{A_i \to$

$B_i\}_{i\in I} \Rightarrow C \vee D$ in $\mathbf{G}$, where Γ is a multiset consisting of Harrop formulas and outputs a $\mathbf{G}$-proof either for $\Gamma, \{A_i \to B_i\}_{i\in I} \Rightarrow C$ or $\Gamma, \{A_i \to B_i\}_{i\in I} \Rightarrow D$ or $\Gamma, \{A_i \to B_i\}_{i\in I} \Rightarrow A_i$, for some $i \in I$. If the algorithm works only for the empty Γ, we say the sequent calculus $\mathbf{G}$ has *feasible Visser property*, or *feasible VP*. If I is empty, we say the sequent calculus $\mathbf{G}$ has *feasible Harrop property*, or *feasible HP* and if both are empty, we say the sequent calculus $\mathbf{G}$ has *feasible disjunction property*, or *feasible DP*.

Theorem 4.12. *Let $\mathbf{G}$ be a sequent calculus extending $i\mathbf{K}$ consisting of almost positive rules, cut and Nec. Then, if $\mathbf{G} \vdash^{\pi} \Gamma \Rightarrow \Delta$ then there exist a multiset Σ_π and a $\mathbf{G}$-proof σ_π, such that $\mathbf{G} \vdash \Sigma_\pi, \Gamma^t \Rightarrow \Delta^t$ and the following conditions hold:*

- *(i) The formulas in Σ_π are modal Horn formulas constructed from the angled atoms;*
- *(ii) $\mathbf{G} \vdash^{\sigma_\pi} \Rightarrow \bigwedge \Sigma_\pi^s$, where s is the standard substitution;*
- *(iii) The process of finding Σ_π and σ_π from π is feasible, i.e., there exists a polynomial time (in the length of π) algorithm that reads the proof π and outputs the multiset Σ_π and the proof σ_π.*

Proof. We first provide the algorithm that produces Σ_π and σ_π. We check the feasibility later. The algorithm computes Σ_π and σ_π by recursion on the structure of π. If the last rule is a left almost positive rule

$$\frac{\{\Gamma, \overline{N'_i(\bar{\phi})} \Rightarrow \overline{M'_i(\bar{\phi})}, \Delta\}_i}{\Gamma, \overline{M(\bar{\phi})} \Rightarrow \Delta}$$

By induction hypothesis, we have $\Sigma', \Gamma^t, \overline{(N'_i(\bar{\phi}))^t} \Rightarrow \overline{(M'_i(\bar{\phi}))^t}, \Delta^t$. Now we have two cases to consider. For the first case, if the formulas in $\overline{N_i}$'s are all basic formulas, we claim that $\Sigma = \Sigma' \cup \bigcup_i \bigcup_{A \in \overline{M'}_i} \Pi_A \cup \bigcup_{A \in \overline{M}} \Pi_A \cup \bigcup_i \bigcup_{B \in \overline{N'}_i} \Phi_B$ works, i.e., $\Sigma, \Gamma^t, \overline{(M(\bar{\phi}))^t} \Rightarrow \Delta^t$. First, by Lemma 4.5, note that $\Pi_A, (A(\bar{\phi}))^t \Rightarrow A(\overline{\phi^t})$, for any $A \in \overline{M}_i$. Hence, $\Sigma, \Gamma^t, \overline{(N'_i(\bar{\phi}))^t} \Rightarrow \overline{M'_i(\overline{\phi^t})}, \Delta^t$. Moreover, by the same lemma, we have $\Phi_B, (B(\bar{\phi}))^t \Rightarrow B(\overline{\phi^t})$, for any $B \in \overline{N'}_i$. Hence, we have $\Sigma, \Gamma^t, \overline{N'_i(\overline{\phi^t})} \Rightarrow \overline{M'_i(\overline{\phi^t})}, \Delta^t$. By the rule itself

$$\frac{\{\Sigma, \Gamma^t, \overline{N'_i(\overline{\phi^t})} \Rightarrow \overline{M'_i(\overline{\phi^t})}, \Delta^t\}_i}{\Sigma, \Gamma^t, \overline{M(\overline{\phi^t})} \Rightarrow \Delta^t}$$

Finally, by Lemma 4.5, since $\Pi_A, (A(\overline{\phi}))^t \Rightarrow A(\overline{\phi^t})$, for any $A \in \overline{M}$, we have $\Sigma, \Gamma^t, \overline{M(\overline{\phi})^t} \Rightarrow \Delta^t$ in $i\mathbf{K}$.
For the second case (the rule is a left almost positive rule and $n \leqslant 1$), we have to address two cases $n = 0$ and $n = 1$. If $n = 0$, by the rule itself we have

$$\frac{}{\Sigma, \Gamma^t, \overline{M(\overline{\phi^t})} \Rightarrow \Delta^t}$$

since $\Pi_A, (A(\overline{\phi}))^t \Rightarrow A(\overline{\phi^t})$, for any $A \in \overline{M}$, we have $\Sigma, \Gamma^t, \overline{(M(\overline{\phi}))^t} \Rightarrow \Delta^t$ which is what we wanted.
If $n = 1$, the last rule is

$$\frac{\Gamma, \{N'_j(\overline{\phi})\}_j \Rightarrow \overline{M'(\overline{\phi})}, \Delta}{\Gamma, \overline{M(\overline{\phi})} \Rightarrow \Delta}$$

We claim that

$$\Sigma = \Sigma' \cup \bigcup_{A \in \overline{M'}} \Pi_A \cup \bigcup_{A \in \overline{M}} \Pi_A \cup \bigcup_{B \in \overline{N'}} \Upsilon_B \cup \bigcup_j \Big(\bigwedge_{A \in \overline{M(\overline{\phi})})} \langle A(\overline{\phi}) \rangle \to \langle N_j(\overline{\phi}) \rangle \Big)$$

works, i.e., $\Sigma, \Gamma^t, \overline{(M(\overline{\phi}))^t} \Rightarrow \Delta^t$. First, note that any formula in Σ is constructed of the angled atoms and is modal Horn. Second, as $\Pi_A, (A(\overline{\phi}))^t \Rightarrow A(\overline{\phi^t})$, for any $A \in \overline{M'}$, we have $\Sigma, \Gamma^t, \{(N'_j(\overline{\phi}))^t\} \Rightarrow \overline{M'(\overline{\phi^t})}, \Delta^t$. Then, by Lemma 4.5, we have

$$\Upsilon_{N'_j}, \langle N'_j(\overline{\phi}) \rangle, N'_j(\overline{\phi^t}) \Rightarrow (N'_j(\overline{\phi}))^t,$$

for any j. Hence, we have

$$\Sigma, \Gamma^t, \{\langle N_j(\overline{\phi}) \rangle, N_j(\overline{\phi^t})\}_j \Rightarrow \overline{M'(\overline{\phi^t})}, \Delta^t$$

By the rule itself

$$\frac{\Sigma, \Gamma^t, \{\langle N'_j(\overline{\phi}) \rangle, N'_j(\overline{\phi^t})\}_j \Rightarrow \overline{M'(\overline{\phi^t})}, \Delta^t}{\Sigma, \Gamma^t, \{\langle N'_j(\overline{\phi}) \rangle\}_j, \overline{M(\overline{\phi^t})} \Rightarrow \Delta^t}$$

Since for any j, the formula $(\bigwedge_{A \in \overline{M(\overline{\phi})})} \langle A(\overline{\phi}) \rangle) \to \langle N'_j(\overline{\phi}) \rangle$ is in Σ, we have

$$\Sigma, \Gamma^t, \bigwedge_{A \in \overline{M(\overline{\phi})})} \langle A(\overline{\phi}) \rangle), \overline{M(\overline{\phi^t})} \Rightarrow \Delta^t$$

Finally, as $(A(\overline{\phi}))^t \Rightarrow \langle A(\overline{\phi}) \rangle$, we have $\overline{(M(\overline{\phi}))^t} \Rightarrow \bigwedge_{A \in \overline{M(\overline{\phi})})} \langle A(\overline{\phi}) \rangle)$. Moreover, by Lemma 4.5, we have $\Pi_{A, \overline{\phi}}, (A(\overline{\phi}))^t \Rightarrow A(\overline{\phi^t})$. Hence, we have

$\Sigma, \Gamma^t, \overline{(M(\overline{\phi}))^t} \Rightarrow \Delta^t$.
Now, it is enough to show that $(\bigwedge_{A \in \overline{M(\overline{\phi})}} \langle A(\overline{\phi}) \rangle)^s \Rightarrow \langle N_j(\overline{\phi}) \rangle^s$ holds in **G**, which is the case, because it is provable by the rule itself

$$\frac{\{N'_j(\overline{\phi})\}_j \Rightarrow \overline{M'(\overline{\phi})}, N'_j(\overline{\phi})}{\overline{M(\overline{\phi})} \Rightarrow N'_j(\overline{\phi})}$$

For the right rule, we also have two cases. For the context-free right rule, if the last rule is

$$\frac{\{\Gamma, \overline{N'_i(\overline{\phi})} \Rightarrow \overline{M'_i(\overline{\phi})}\}_i}{\Gamma \Rightarrow \overline{N(\overline{\phi})}} \; ,$$

by induction hypothesis, we have $\Sigma', \Gamma^t, \overline{(N'_i(\overline{\phi}))^t} \Rightarrow \overline{(M'_i(\overline{\phi}))^t}$. Now we have two cases to consider. For the first case, if the formulas in $\overline{N}$'s are all basic formulas, we claim that $\Sigma = \Sigma' \cup \bigcup_i \bigcup_{A \in \overline{M'}_i} \Pi_A \cup \bigcup_{A \in \overline{N}} \Phi_A \cup \bigcup_i \bigcup_{B \in \overline{N'_i}} \Phi_B$ works, i.e., $\Sigma, \Gamma^t \Rightarrow \overline{(N(\overline{\phi}))^t}$. First, by Lemma 4.5, we have $\Pi_A, (A(\overline{\phi}))^t \Rightarrow A(\overline{\phi^t})$, for any $A \in \overline{M}_i$. Hence, $\Sigma, \Gamma^t, \overline{(N'_i(\overline{\phi}))^t} \Rightarrow \overline{M'_i(\overline{\phi^t})}, \Delta^t$. Moreover, by the same lemma, we have $\Phi_B, (B(\overline{\phi}))^t \Rightarrow B(\overline{\phi^t})$, for any $B \in \overline{N'}_i \cup \overline{N}$. Hence, we have $\Sigma, \Gamma^t, \overline{N'_i(\overline{\phi^t})} \Rightarrow \overline{M'_i(\overline{\phi^t})}$. Finally, by the rule itself

$$\frac{\Sigma, \Gamma^t, \overline{N'_i(\overline{\phi^t})} \Rightarrow \overline{M'_i(\overline{\phi^t})}}{\Sigma, \Gamma^t \Rightarrow \overline{N(\overline{\phi^t})}}$$

and by Lemma 4.5, we have $\Sigma, \Gamma^t \Rightarrow \overline{(N(\overline{\phi}))^t}$.
For the other case, if $\overline{N}$ is just one almost negative formula, we claim that

$$\Sigma = \Sigma' \cup \bigcup_i \bigcup_{A \in \overline{M'}_i} \Pi_A \cup \Upsilon_N \cup \bigcup_i \bigcup_{B \in \overline{N'_i}} \Phi_B \cup \{\bigwedge_{\gamma \in \Gamma} \langle \gamma \rangle \to \langle N(\overline{\phi}) \rangle\}$$

works. With the same line of argument as before, we have $\Sigma, \Gamma^t \Rightarrow N(\overline{\phi^t})$. Then, by Lemma 4.5, we have

$$\Upsilon_N, \langle N(\overline{\phi}) \rangle, N(\overline{\phi^t}) \Rightarrow (N(\overline{\phi}))^t$$

Since $\bigwedge_{\gamma \in \Gamma} \langle \gamma \rangle \to \langle N(\overline{\phi}) \rangle \in \Sigma$, we have $\Sigma, \Gamma^t, \bigwedge_{\gamma \in \Gamma} \langle \gamma \rangle \Rightarrow (N(\overline{\phi}))^t$. Finally, as $\Gamma^t \Rightarrow \langle \gamma \rangle$, we have $\Sigma, \Gamma^t \Rightarrow (N(\overline{\phi}))^t$.
Therefore, it is enough to show $(\bigwedge_{\gamma \in \Gamma} \langle \gamma \rangle)^s \Rightarrow \langle N(\overline{\phi}) \rangle^s$ which is provable as we have $\Gamma \Rightarrow N(\overline{\phi})$.
For the contextual right rule, if the last rule is:

$$\frac{\{\Gamma \Rightarrow \overline{M'_i(\overline{\phi})}, \Delta\}_{i=1}^{n}}{\Gamma \Rightarrow \overline{N(\overline{\phi})}, \Delta}$$

By induction hypothesis, we have $\Sigma', \Gamma^t \Rightarrow \overline{(M'_i(\overline{\phi}))^t}, \Delta^t$. Again we have two cases. If all formulas in $\overline{N}$ are basic, set

$$\Sigma = \Sigma' \cup \bigcup_i \bigcup_{A \in \overline{M'}_i} \Pi_A \cup \bigcup_i \bigcup_{B \in \overline{N}} \Phi_B.$$

The proof is similar to that of the previous case. If $\overline{N}$ is just one formula, then we claim that

$$\Sigma = \Sigma' \cup \bigcup_i \bigcup_{A \in \overline{M'}_i} \Pi_A \cup \Upsilon_N \cup \{\bigwedge_i \langle A_{f(i)} \rangle \to \langle N(\overline{\phi}) \rangle\}_{f \in X}$$

works, where $X = \{f : \{1, \ldots, n\} \to \bigcup_i \bigcup \overline{M'_i} \mid \forall i \; f(i) \in \overline{M'_i}\}$. Note that the number of elements in X is constant ($|X| = \binom{\Sigma_{i=1}^{n} |\overline{M'_i}|}{n}$). The reason is that for the given rule, the number of premises and the number of elements in $\overline{M'_i}$ are determined and fixed. By Lemma 4.5, since $\Pi_A, A(\overline{\phi})^t \Rightarrow A(\overline{\phi}^t)$, for any $A \in \overline{M}_i$, we have $\Gamma^t \Rightarrow \overline{M'_i(\overline{\phi}^t)}, \Delta^t$. Now, by applying the rule we have

$$\frac{\{\Sigma', \Gamma^t \Rightarrow \overline{M'_i(\overline{\phi^t})}, \Delta^t\}_i}{\Sigma', \Gamma^t \Rightarrow N(\overline{\phi^t}), \Delta^t}$$

To complete the proof we have to show that $\Sigma, \Gamma^t \Rightarrow \langle N(\overline{\phi}) \rangle, \Delta^t$. Since $(A(\overline{\phi}))^t \Rightarrow \langle A(\overline{\phi}) \rangle$, for any i and $A \in \bigcup_i \overline{M'_i}$, we have $\Sigma, \Gamma^t \Rightarrow \bigwedge_i \bigvee_{A \in \overline{M'_i}} A, \Delta$ that by distributivity implies $\Sigma, \Gamma^t \Rightarrow \bigvee_{f \in X} \bigwedge_i f(i), \Delta$. Therefore, it is enough to prove that for any $f \in X$, we have $\Sigma, \Gamma^t, \bigwedge_i \langle A_{f(i)} \rangle \Rightarrow \langle N(\overline{\phi}) \rangle, \Delta$ which is trivially true as $\bigwedge_i \langle A_{f(i)} \rangle \to \langle N(\overline{\phi}) \rangle$ is in Σ. Finally, we have to show that $(\bigwedge_i \langle A_{f(i)} \rangle)^s \Rightarrow \langle N(\overline{\phi}) \rangle)^s$. This is provable by the rule

$$\frac{\{\bigwedge_i A_{f(i)} \Rightarrow \overline{M'_i(\overline{\phi})}\}_i}{\bigwedge_i A_{f(i)} \Rightarrow N(\overline{\phi})}$$

If the last rule used in the proof is Nec:

$$\frac{\Rightarrow A}{\Rightarrow \Box A}$$

By induction hypothesis, we have $\Sigma' \Rightarrow A^t$. Using the rule $K_\Box$, we have $\Box \Sigma' \Rightarrow \Box A^t$. Set $\Sigma = \Box \Sigma' \cup \{\langle \Box A \rangle\}$. It is clear that $\Sigma \Rightarrow (\Box A)^t$. The case where the last rule used in the proof is the cut rule, is similar. We can easily show the feasibility of the process of finding Σ_π and σ_π from π, similar to the analysis of the feasibility in Theorem 4.5. □

Definition 4.13. A calculus $\mathbf{G}$ is called *T-free* if it is valid in the irreflexive Kripke frame of one node. It is called *T-full* if it is valid in the reflexive Kripke frame of one node and extends $i\mathbf{K} + T_a + T_b$. A calculus over the languages $\mathcal{L}_\Box$, $\mathcal{L}_\Diamond$ and $\mathcal{L}_p$ is called T-free if it is T-free as a calculus over the languages $\mathcal{L}_\Box$, $\mathcal{L}_\Diamond$ and $\mathcal{L}_p$. A calculus over the languages $\mathcal{L}_\Box$, $\mathcal{L}_\Diamond$ and $\mathcal{L}_p$ is called T-full if it is valid in the reflexive Kripke frame of one node and extends $i\mathbf{K}_\Box + T_a$, $i\mathbf{K}_\Diamond + T_b$, and $\mathbf{LJ}$, respectively.

Lemma 4.14. *Let $\mathbf{G}$ be a T-free or a T-full calculus. Then, there is a feasible algorithm reading a $\mathbf{G}$-proof of a sequent $\Sigma, \{\neg q_j\}_{j=1}^m \Rightarrow \{p_i\}_{i=1}^n$ where Σ is a multiset of modal Horn formulas and p_i's and q_j's are atomic formulas to compute a multiset Σ' consisting of implicational Horn formulas and a $\mathbf{G}$-proof π such that:*

- $\mathbf{LK} \vdash \Sigma' \Rightarrow \{p_i\}_{i=1}^n, \{q_j\}_{j=1}^m$,
- $\mathbf{G} \vdash^\pi \Sigma \Rightarrow \bigwedge \Sigma'$.

Proof. If $\mathbf{G}$ is T-free, set Σ' as the set of modality-free formulas in Σ. Note that the process of computing this multisets is polynomial time in the length of $\Sigma, \{\neg q_j\}_{j=1}^m \Rightarrow \{p_i\}_{i=1}^n$. To show $\mathbf{LK} \vdash \Sigma' \Rightarrow \{p_i\}_{i=1}^n, \{q_j\}_{j=1}^m$, if $\mathbf{LK} \nvdash \Sigma' \Rightarrow \{p_i\}_{i=1}^n, \{q_j\}_{j=1}^m$, there exists a classical model I such that $I \vDash \Sigma'$ but $I \nvDash p_i$ and $I \nvDash q_j$ for any $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. We use this model to build a Kripke model for $\mathbf{G}$ in the following manner: Define the Kripke frame as the singleton set $\{w\}$ with the empty accessibility relation. Therefore, w is irreflexive. Define the valuation function as $V(p) = I(p)$. It is easy to see that for any modality-free formula ϕ, $w \vDash \phi$ if and only if $I(\phi) = 1$. Hence, $w \vDash \Sigma'$. On the other hand, since the node w is irreflexive, for any formula C, $w \vDash \Box C$, and $w \nVdash \Diamond C$. Therefore, for any modal-Horn formula D in the form $\Diamond^m q \to r$ or $q \to \Box^n r$ or $\Box^k C$, for some $m, n, k \geqslant 1$, we have $w \vDash D$. Hence, $w \vDash \Sigma$. However, the atoms p_i and q_j are not satisfied in this model, while $\mathbf{G}$ is valid in any model based on the irreflexive node which is a contradiction. Hence, $\mathbf{LK} \vdash \Sigma' \Rightarrow \{p_i\}_{i=1}^n, \{q_j\}_{j=1}^m$. The other point is clear as $\Sigma' \subseteq \Sigma$.

For the other case, if $\mathbf{G}$ is T-full, define the new multiset Σ' as Σ but delete every $\Box$ and $\Diamond$ in any of their formulas. Hence, formulas in Σ' are in implicational Horn form. Suppose $\mathbf{LK} \nvdash \Sigma' \Rightarrow \{p_i\}_{i=1}^n, \{q_j\}_{j=1}^m$. Then, again there exists a classical model I such that $I \vDash \Sigma'$ and $I \nvDash p_i$, and $I \nvDash q_j$, for any $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. We use this I to build a Kripke model in the following way: Define the Kripke frame as the singleton set $\{w\}$ with the accessibility relation $\{(w, w)\}$. Therefore, w is reflexive. Define the valuation function as $V(p) = I(p)$. Since the model is only a reflexive node, the modality collapses and hence the validity of Σ' implies the validity of Σ in w. Therefore, $w \vDash \Sigma$ but $w \nvDash p_i$ and $w \nvDash q_j$, for any $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Since $\mathbf{G}$ is valid in any model based on the reflexive node, we reach a contradiction. Hence, $\mathbf{LK} \vdash \Sigma \Rightarrow \{p_i\}_{i=1}^n, \{q_j\}_{j=1}^m$. For the other point, note that $\mathbf{G}$ is T-full and hence it extends $i\mathbf{K} + T_a + T_b$. Therefore, it proves $\Box C \Rightarrow C$ and $C \Rightarrow \Diamond C$, for any formula C. It implies that for any modal Horn formula $\Diamond^m q \to r$ or $q \to \Box^n r$ or $\Box^k C$, we have $\mathbf{G} \vdash D \Rightarrow D'$ in a feasible way, where D' is D after eliminating all the modalities. Hence, $\mathbf{G} \vdash \Sigma \Rightarrow \bigwedge \Sigma'$. The feasibility part is again simple and similar to the feasibility in Theorems 4.5 and 4.12. $\Box$

Now we are ready to state our main result:

Theorem 4.15. *Let $\mathbf{G}$ be a T-free or a T-full calculus consisting only of positive rules, the cut rule and Nec and extending $i\mathbf{K}$. Then, $\mathbf{G}$ has feasible Visser-Harrop property.*

Proof. Suppose two formulas C and D, a multiset Γ of Harrop formulas, a multiset of implications $\{A_i \to B_i\}_{i \in I}$ and a proof π are given such that $\mathbf{G} \vdash^{\pi} \Gamma, \{A_i \to B_i\}_{i \in I} \Rightarrow C \vee D$. By Theorem 4.12, we can get a multiset Σ_π of modal Horn formulas constructed only from the angled atoms such that $\mathbf{G} \vdash \Sigma_\pi, \Gamma^t \{(A_i \to B_i)^t\}_{i \in I} \Rightarrow (C \vee D)^t$. Using the $i\mathbf{K}$-provable sequents $A_i^t \to \bot \Rightarrow A_i^t \to B_i^t$ and $\langle A_i \to B_i \rangle, A_i^t \to \bot \Rightarrow (A_i \to B_i)^t$, we have $\mathbf{G} \vdash \Sigma_\pi, \Gamma^t, \{\langle A_i \to B_i \rangle\}_{i \in I}, \{\neg A_i^t\}_{i \in I} \Rightarrow C^t, D^t$ which by Lemma 4.3 implies $\mathbf{G} \vdash \Sigma_\pi, \Gamma^t, \{\langle A_i \to B_i \rangle\}_{i \in I}, \{\neg \langle A_i \rangle\}_{i \in I} \Rightarrow \langle C \rangle, \langle D \rangle$. Use Lemma 4.10 to find a multiset Λ of modal-Horn formulas built from angled atoms such that the standard substitution of Λ is equivalent to Γ in $i\mathbf{K}$ and hence in $\mathbf{G}$. Then, $\mathbf{G} \vdash \Sigma_\pi, \Lambda, \{\langle A_i \to B_i \rangle\}_{i \in I}, \{\neg \langle A_i \rangle\}_{i \in I} \Rightarrow \langle C \rangle, \langle D \rangle$. By Lemma 4.14, we can feasibly provide a multiset Ω such that $\mathbf{LK} \vdash \Omega, \{\neg \langle A_i \rangle\}_{i \in I} \Rightarrow \langle C \rangle, \langle D \rangle$, and $\mathbf{G} \vdash^{\sigma} \Sigma_\pi, \Lambda, \{\langle A_i \to B_i \rangle\}_{i \in I} \Rightarrow \bigwedge \Omega$. Take $S = \Omega \Rightarrow \langle C \rangle, \langle D \rangle, \{\langle A_i \rangle\}_{i \in I}$. We can use Lemma 4.7 to find τ and $1 \leqslant i \leqslant n$, such that $\mathbf{LJ} \vdash^{\tau} \Omega \Rightarrow \langle A_i \rangle$ or $\mathbf{LJ} \vdash^{\tau} \Omega \Rightarrow \langle C \rangle$ or $\mathbf{LJ} \vdash^{\tau} \Omega \Rightarrow \langle D \rangle$. For simplicity, assume $\mathbf{LJ} \vdash^{\tau} \Omega \Rightarrow \langle C \rangle$. The rest of the cases are the same. Since $\mathbf{G}$ extends $\mathbf{LJ}$, we also have $\mathbf{G} \vdash \Omega \Rightarrow \langle C \rangle$. Using the fact that $\mathbf{G} \vdash^{\sigma} \Sigma_\pi, \Lambda, \{\langle A_i \to B_i \rangle\}_{i \in I} \Rightarrow \bigwedge \Omega$ we have $\mathbf{G} \vdash \Sigma_\pi, \Lambda, \{\langle A_i \to B_i \rangle\}_{i \in I} \Rightarrow \langle C \rangle$ by a feasible proof in π. The sequent will be provable for any substitution, specially the standard substitution. By Lemma 4.12 we have $i\mathbf{K} \vdash \Rightarrow \bigwedge \Sigma_\pi^s$. Moreover, Lemma 4.10 states that the standard substitution of the conjunction of Λ is equivalent to Γ, feasibly provably in $i\mathbf{K}$. Hence, we have $\mathbf{G} \vdash \Gamma, \{A_i \to B_i\}_{i \in I} \Rightarrow C$. Note that all the steps of the argument are implemented feasibly and hence the whole algorithm is also feasible. $\Box$

Corollary 4.16. *The usual proof systems for* **IK***,* **IKT***,* **IKB***,* **IK4***,* **IK5***,* **IKBT***,* **IS4***,* **IKB4***,* **IK45***,* **IS5***, their Fisher Servei versions and their* $\Diamond$*-free counterparts have feasible Visser-Harrop property and hence feasible DP.*

Corollary 4.17. *(Negative application) Let* $i\mathbf{K} \subseteq L$ *be a logic without disjunction property. Then,* L *has no calculus consisting of almost positive rules and the cut rule and Nec.*

4.1 $\Diamond$-free Fragments

Theorem 4.18. *Let* $\mathbf{G}$ *be a* T*-free or a* T*-full calculus only consisting of almost positive rules, the cut rule and* Nec *and extending* $i\mathbf{K}_{\Box}$*. Then,* $\mathbf{G}$ *has feasible Visser-Harrop property.*

Proof. Let $\mathbf{G}$ be a T-free or a T-full calculus over the language $\mathcal{L}_{\Box} = \{\wedge, \vee, \to, \top, \bot, \Box\}$. The technique is to pretend that $\mathbf{G}$ is a calculus over the extended language $\mathcal{L}$ to apply Theorem 4.15 and then by collapsing $\Diamond$, we can transfer the result to $\mathbf{G}$ itself. More precisely, define $\mathbf{G}^*$ as $\mathbf{G}$ plus the rule $K_{\Diamond}$. (If $\mathbf{G}$ is T-full, also add the rule T_b) over the language $\mathcal{L}$. First, note that $\mathbf{G}^*$ extends $i\mathbf{K}$ (in the T-full case it extends $i\mathbf{K}+T_a+T_b$) and it is clearly T-free or T-full. Second, notice that $\mathbf{G}^*$ is feasibly conservative over $\mathbf{G}$. To see this, it is enough to pick a proof in $\mathbf{G}^*$ for a $\Diamond$-free formula and substitute all $\Diamond A$ in the proof by $\top$. As this substitution transforms the rule $K_{\Diamond}$ (and T_b) to a valid rule in $i\mathbf{K}_{\Box}$, we get a $\mathbf{G}$-proof for the same sequent in polynomial time. Now, we can apply Theorem 4.15 on $\mathbf{G}^*$ to prove the feasible Visser-Harrop property for $\mathbf{G}^*$. However, as $\mathbf{G}^*$ is feasibly conservative over $\mathbf{G}$, this implies the feasible Visser-Harrop property for $\mathbf{G}$. $\Box$

Corollary 4.19. *The usual proof systems for* **IK***,* **IKT***,* **IKB***,* **IK4***,* **IK5***,* **IKBT***,* **IS4***,* **IKB4***,* **IK45***,* **IS5** *have feasible Visser-Harrop property and hence feasible DP.*

Corollary 4.20. *(Negative application) Let* $i\mathbf{K} \subseteq L$ *be a logic without disjunction property. Then,* L *has no calculus consisting of almost positive rules and the cut rule and Nec.*

4.2 $\Box$-free Fragments

Theorem 4.21. *Let* $\mathbf{G}$ *be a* T*-full calculus consisting only of almost positive rules and the cut rule extending* **BLL***. Then,* $\mathbf{G}$ *has feasible Visser-Harrop property.*

Proof. Let **G** be a T-full calculus over the language $\mathcal{L}_\Diamond = \{\wedge, \vee, \to, \top, \bot, \Diamond\}$. The technique is again to pretend that **G** is a calculus over the extended language $\mathcal{L}$ to apply Theorem 4.15 and then by collapsing $\Box$, we can transfer the result to **G** itself. More precisely, define $\mathbf{G}^*$ as **G** plus the rule $K_\Box$, $K_\Diamond$ and the rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Box A} \quad \frac{\Gamma \Rightarrow \Box A}{\Gamma \Rightarrow A}$$

over the language $\mathcal{L}$. First, note that $\mathbf{G}^*$ extends $i\mathbf{K}$ and it is clearly T-full. Second, notice that $\mathbf{G}^*$ is feasibly conservative over **G**. It is just enough to pick a proof in $\mathbf{G}^*$ for a $\Box$-free formula and substitute $\Box A$ by A. As this substitution transforms the rule $K_\Box$ and $K_\Diamond$ to valid rules in BLL, we get a **G**-proof for the same sequent in polynomial time. Now, we can apply Theorem 4.15 on $\mathbf{G}^*$ to prove the feasible Visser-Harrop property for $\mathbf{G}^*$. However, as $\mathbf{G}^*$ is feasibly conservative over **G**, this implies the feasible Visser-Harrop property for **G**. □

Corollary 4.22. *The sequent calculus for the propositional lax logic, defined in Preliminaries, has feasible Visser-Harrop property and hence feasible DP.*

Corollary 4.23. *(Negative application) Let* $\mathsf{BLL} \subseteq L$ *be a* T*-full logic without disjunction property. Then,* L *has no calculus consisting of almost positive rules and the cut rule.*

4.3 Propositional Fragment

Theorem 4.24. *Let* **G** *be a calculus for a superintuitionistic logic only consisting of almost positive rules and the cut rule . Then,* **G** *has feasible Visser-Harrop property.*

Proof. Define $\mathbf{G}^*$ as **G** plus the rules $K_\Box$ and $K_\Diamond$ over the language $\mathcal{L}$. It is clear that **G** extends $i\mathbf{K}$. The calculus $\mathbf{G}^*$ is valid in an irreflexive node, as **G** capture an intermediate logic and hence is valid in the one node Kripke model, specially the irreflexive one. Hence, $\mathbf{G}^*$ is T-free. Finally, it is enough to note that $\mathbf{G}^*$ is feasibly conservative over **G**; for that matter, for any formula A, we substitute each $\Box A$ and $\Diamond A$ by A. □

Corollary 4.25. **LJ** *has the feasible Visser-Harrop property and hence feasible DP.*

Corollary 4.26. *(Negative application) Let* L *be a superintuitionistic logic without disjunction property. Then,* L *has no calculus consisting of almost positive rules and the cut rule.*

5 The Cut-free Case

Definition 5.1. A sequent is called a *strongly focused* axiom if it has one of the following forms:

(1) $\Gamma, \phi \Rightarrow \phi, \Delta$

(2) $\Gamma, \overline{\phi} \Rightarrow \Delta$

(3) $\Gamma \Rightarrow \overline{\phi}, \Delta$

where Γ and Δ are multiset variables, in (3), the formulas in $\overline{\phi}$ have no variables and in (2), any two formulas in $\overline{\phi}$ have the same variables. A rule is called **LJ**-*like* if it has one of the forms right context-free $(r.cf)$, or right contextual $(r.c)$, or a left rule (l):

$$\frac{\{\Gamma, \overline{\phi}_i \Rightarrow \overline{\psi}_i\}_i}{\Gamma \Rightarrow \phi}\,(r.cf) \qquad \frac{\{\Gamma \Rightarrow \overline{\phi}_i, \Delta\}_i}{\Gamma \Rightarrow \phi, \Delta}\,(r.c) \qquad \frac{\{\Gamma, \overline{\phi}_i \Rightarrow \Delta\}_i \qquad \{\Gamma \Rightarrow \overline{\psi}_j\}_j}{\Gamma, \phi \Rightarrow \Delta}\,(l)$$

where in the left rule $|\Delta| \leqslant 1$.

Example 5.2. All the axioms of **LJ** are strongly focused and all of its rules are **LJ**-like. Therefore, IPC clearly has a calculus consisting of strongly focused axioms and **LJ**-like rules. An example of an axiom which is not strongly focused is $(\Gamma \Rightarrow \phi, \neg\phi, \Delta)$, since otherwise it would have been an instance of (3), which is not possible. The reason is that ϕ can have a variable, and by definition, in a sequent of the form (3) formulas in the succedent must be variable free. An example of a rule that is not **LJ**-like is the left and right implication rules in **LK**.

Definition 5.3. Let **G** be a sequent calculus. **G** has the *disjunctive interpolation property*, if for any k and any sequent $S = (\Sigma, \Pi \Rightarrow \Lambda_1, \cdots, \Lambda_k)$, if S is provable in **G**, there exist formulas C_r, for $1 \leqslant r \leqslant k$ such that $\Sigma \Rightarrow C_1, \cdots C_k$ and $(\Pi, C_r \Rightarrow \Lambda_r)$ are provable in **G** and $V(C_r) \subseteq V(\Sigma) \cap V(\Pi \Rightarrow \Lambda_r)$, where $V(A)$ is the set of the atoms of A.

Theorem 5.4. *Let* **G** *be a sequent calculus for a superintuitionistic logic L. If* **G** *has the disjunctive interpolation property, then L has both the Craig interpolation property and disjunction property.*

Proof. For the Craig interpolation property, assume $A \to B \in L$. Then, $\mathbf{G} \vdash A \to B$. Set $\Sigma = A$, $\Pi = \varnothing$, $k = 1$ and $\Lambda_1 = B$. Then, there exists a formula C_1 such that $\mathbf{G} \vdash A \Rightarrow C$ and $\mathbf{G} \vdash C \Rightarrow B$ and $V(C) \subseteq V(A) \cap V(B)$. Therefore, $A \to C, C \to B \in L$.

For the disjunction property, assume $A \vee B \in L$. Then, $\mathbf{G} \vdash \Rightarrow A, B$. Set $\Sigma = \Pi = \varnothing$, $k = 2$ and $\Lambda_1 = A$ and $\Lambda_2 = B$. Then, there are C_1 and C_2 such that $\mathbf{G} \vdash \Rightarrow C_1, C_2$, $\mathbf{G} \vdash C_1 \Rightarrow A$, $\mathbf{G} \vdash C_2 \Rightarrow B$ and $V(C_1) = V(C_2) = \varnothing$. As $\mathbf{G}$ extends $\mathbf{LJ}$ and in $\mathbf{LJ}$, any variable-free formula is equivalent to $\top$ or $\bot$, the formulas C_1 and C_2 are equivalent to $\top$ or $\bot$ in $\mathbf{G}$. As $\mathbf{G} \vdash \Rightarrow C_1, C_2$, we have $C_1 \vee C_2 \in \mathsf{CPC}$. Therefore, at least one of C_1 and C_2 is equivalent to $\top$. Therefore, by $\mathbf{G} \vdash C_1 \Rightarrow A$, $\mathbf{G} \vdash C_2 \Rightarrow B$ and the fact that $\mathsf{IPC} \subseteq L$ and the closure of L under modus ponens we have either $A \in L$ or $B \in L$. $\square$

Theorem 5.5. *(Disjunctive interpolation) Let* $\mathbf{G}$ *be a sequent calculus consisting of strongly focused axioms and* $\mathbf{LJ}$*-like rules, extending* $\mathbf{LJ}$*. Then,* $\mathbf{G}$ *has the disjunctive interpolation property.*

Proof. The proof uses induction. For axioms, we will consider the strongly focused axioms one by one:

(1) In this case the sequent S is of the form $(\Gamma, \phi \Rightarrow \phi, \Delta)$. Let $\phi \in \Lambda_1$. There are two cases to consider. If $\phi \in \Sigma$, then set $C_1 = \phi$ and $C_r = \bot$, for $r \neq 1$. By the axiom itself $\Sigma \Rightarrow C_1, C_2, \cdots, C_k$. Moreover, as $C_r = \bot$, we have $\Pi, C_r \Rightarrow \Lambda_r$, for $r \neq 1$. For $r = 1$, as $\phi \in \Lambda_1$, we have $\Pi, C_1 \Rightarrow \Lambda_1$. For the variable condition, as $V(C_r) = \varnothing$, for $r \neq 1$, the only thing to check is that $V(C_1) \subseteq V(\Sigma) \cap V(\Pi \Rightarrow \Lambda_1)$ which is clear. If $\phi \in \Pi$, then set $C_1 = \top$ and $C_r = \bot$, for $r \neq 1$. It is clear that $\Sigma \Rightarrow C_1, C_2, \cdots, C_k$ and $\Pi, C_r \Rightarrow \Lambda_r$, for $r \neq 1$. For $r = 1$, as $\phi \in \Lambda_1 \cap \Pi$, we have $\Pi, C_1 \Rightarrow \Lambda_1$. As $V(C_r) = \varnothing$, for any r, there is nothing to check for variable conditions.

(2) If S is of the form $\Gamma, \overline{\phi} \Rightarrow \Delta$ define $C_r = \bot$. First, note that we have $\Sigma, \overline{\phi} \Rightarrow \bot, \bot, \cdots, \bot$, where in the succedent we have k many $\bot$'s. The reason is that this sequent is an instance of the axiom (2) itself. Moreover, for every r we have $\Pi, \bot \Rightarrow \Lambda_r$ since it is an instance of the axiom $\bot$. And again $V(C_r) = \varnothing$.

(3) If S is of the form $(\Gamma \Rightarrow \overline{\phi}, \Delta)$ define $C_r = \bigvee(\Lambda_r \cap \overline{\phi})$. It is easy to see that this C_r works. Because, $\Pi, C_r \Rightarrow \Lambda_r$ is an instance of an axiom. We also have $\Sigma \Rightarrow C_1, \cdots, C_k$, since in the succedent we will have the formula $\overline{\phi}$ (together with some other formulas which we will treat as the context) and it will become an instance of the same axiom. Note that since $V(\overline{\phi}) = \varnothing$, there is nothing to check for the variables.

For the rules, there are three cases to consider based on the last rule of the proof.

- If the last rule used in the proof is a right context-free **LJ**-like rule, then it is of the following form:

$$\frac{\{\Gamma, \overline{\phi}_i \Rightarrow \overline{\psi}_i\}_i}{\Gamma \Rightarrow \phi}$$

By induction hypothesis we have $\Sigma \Rightarrow D_i$ and $\Pi, D_i, \overline{\phi}_i \Rightarrow \overline{\psi}_i$. Define $C = \bigwedge_i D_i$. We have $\Sigma \Rightarrow C$ and $\Pi, C, \overline{\phi}_i \Rightarrow \overline{\psi}_i$. Therefore, by the rule itself, we have $\Pi, C \Rightarrow \phi$.

- If the last rule used in the proof is a right contextual **LJ**-like rule, then it is of the following form:

$$\frac{\{\Gamma \Rightarrow \overline{\phi}_i, \Delta\}_i}{\Gamma \Rightarrow \phi, \Delta}$$

where $\Lambda_1, \cdots, \Lambda_k$ are given such that $\bigcup_{j=1}^{k} \Lambda_j = \Delta \cup \{\phi\}$. W.l.o.g. suppose $\phi \in \Lambda_1$ and we denote $\Lambda_1 - \{\phi\}$ by Λ'_1. By induction hypothesis we have $\Sigma \Rightarrow D_{i1}, D_{i2} \cdots, D_{ik}$ and $\Pi, D_{i1} \Rightarrow \overline{\phi}_i, \Lambda'_1$ and for any $r \neq 1$ we have $\Pi, D_{ir} \Rightarrow \Lambda_r$. Define $C_1 = \bigwedge_i D_{i1}$ and for any $r \neq 1$, define $C_r = \bigvee_i D_{ir}$. For any $r \neq 1$, it is clear that $\Pi, C_r \Rightarrow \Lambda_r$. For $r = 1$, as $\Pi, C_1 \Rightarrow \overline{\phi}_i, \Lambda'_1$, by the rule itself we have $\Pi, C_1 \Rightarrow \phi, \Lambda'_1$. Finally, as $\Gamma \Rightarrow D_{i1}, D_{i2} \cdots, D_{ik}$, we have $\Gamma \Rightarrow D_{i1}, C_2 \cdots, C_k$. Hence, $\Gamma \Rightarrow C_1, C_2 \cdots, C_k$.

- If the last rule used in the proof is a left **LJ**-like rule, then it is of the form:

$$\frac{\{\Gamma, \overline{\phi}_i \Rightarrow \Delta\}_i \qquad \{\Gamma \Rightarrow \overline{\psi}_j\}_j}{\Gamma, \phi \Rightarrow \Delta}$$

There are two cases to consider. First, if $\phi \in \Pi$. Set $\Pi' = \Pi - \{\phi\}$. Then, by induction hypothesis there are formulas D_i and E_j such that

$$\Sigma \Rightarrow D_i \quad , \quad \Pi', \overline{\phi}_i, D_i \Rightarrow \Delta$$

$$\Sigma \Rightarrow E_j \quad , \quad \Pi', E_j \Rightarrow \overline{\psi}_j$$

Define $C = \bigwedge_i D_i \wedge \bigwedge_j E_j$. Then, we have $\Sigma \Rightarrow C$ and $\Pi', C \Rightarrow \overline{\psi}_j$ and $\Pi', C, \overline{\phi}_i \Rightarrow \Delta$. By the rule itself, we have $\Pi', \phi, C \Rightarrow \Delta$.
If $\phi \in \Sigma$, then set $\Sigma' = \Sigma - \{\phi\}$. By IH, there are formulas D_i and E_j such that

$$\Pi, D_i \Rightarrow \Delta \quad , \quad \Sigma', \overline{\phi}_i \Rightarrow D_i$$

$$\Pi \Rightarrow E_j \quad , \quad \Sigma', E_j \Rightarrow \overline{\psi}_j$$

Set $C = \bigwedge_j E_j \to \bigvee_i D_i$. We have $\Sigma', \bigwedge_j E_j \Rightarrow \overline{\psi}_j$ and $\Sigma', \overline{\phi}_i, \bigwedge_j E_j \Rightarrow \bigvee_i D_i$. Hence, by the rule $\Sigma', \phi, \bigwedge_j E_j \Rightarrow \bigvee_i D_i$ that implies $\Sigma', \phi \Rightarrow C$. Moreover, as $\Pi, \bigwedge_j E_j \to \bigvee_i D_i \Rightarrow \Delta$, we have $\Pi, C \Rightarrow \Delta$.

□

Corollary 5.6. *(Characterization Theorem for* IPC*) A superintutionistic logic L has a calculus only consisting of strongly focused axioms and* **LJ***-like rules iff $L =$* IPC*.*

Proof. It is a clear consequence of Theorem 5.5 and the fact that IPC is the only superintuitionistic logic with both Craig interpolation and disjunction property. □

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