## On the Existence of Proof Systems

Amirhossein Akbar Tabatabai

Department of Philosophy, Utrecht University

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A. Akbar Tabatabai

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- Proposing a convincing formalization of what we mean by nice proof systems,
- Finding an invariant, i.e., a property that the logic of a nice proof system enjoys,
- And finally, proving that the property is **rare**, i.e., almost all logics in the family do not enjoy the property.

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  - Then, we connect the form of the rules to some variants of interpolation property of the logic that the system captures.
  - As interpolation is a **rare** property, we prove that nice proof systems are rare.

• Left semi-analytic rule:

$$\frac{\langle \langle \Pi_j, \bar{\psi}_{j\mathfrak{s}} \Rightarrow \bar{\theta}_{j\mathfrak{s}} \rangle_{\mathfrak{s}} \rangle_j}{\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

where  $\Pi_i$ ,  $\Gamma_i$  and  $\Delta_i$ 's are meta-multiset variables and

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where  $\Pi_j$ ,  $\Gamma_i$  and  $\Delta_i$ 's are meta-multiset variables and  $\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi).$  • Left semi-analytic rule:

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## Example

$$\begin{array}{c} \overline{\Gamma, \phi \Rightarrow \Delta} & \overline{\Gamma, \psi \Rightarrow \Delta} \\ \overline{\Gamma, \phi \lor \psi \Rightarrow \Delta} & \overline{\Gamma_1, \phi \Rightarrow \Delta_1} & \overline{\Gamma_2, \psi \Rightarrow \Delta_2} \\ \hline \overline{\Gamma_1, \Gamma_2, \phi + \psi \Rightarrow \Delta_1, \Delta_2} \\ \\ \hline \frac{\Pi \Rightarrow \phi}{\Gamma, \Pi, \phi \to \psi \Rightarrow \Delta} \end{array}$$

• Right semi-analytic rule:

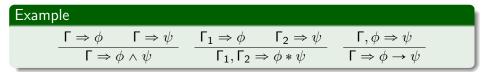
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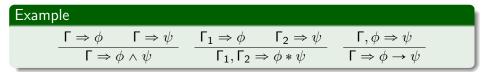
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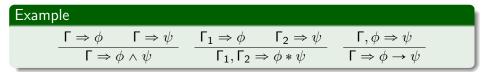
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• Left multi-conclusion semi-analytic rule:  $\frac{\langle \langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$ 

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#### Example

The following rules are semi-analytic:

- The usual conjunction, disjunction and implication rules for IPC;
- All the rules in sub-structural logic FL<sub>e</sub>, weakening and contraction rules;
- The following rules for exponentials in linear logic:

$$\frac{\Gamma, !\phi, !\phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

#### Example

- The cut rule; since it does not meet the variable occurrence condition.
- the following rule in the calculus of **KC**:

$$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}$$

in which  $\Delta$  should consist of negation formulas is not a multi-conclusion semi-analytic rule, simply because the context is not free for all possible substitutions.

## Focused axioms

A sequent is called a *focused axiom* if it has the following form:

- (1)  $(\phi \Rightarrow \phi)$
- (2)  $(\Rightarrow \bar{\alpha})$
- (3)  $(\bar{\beta} \Rightarrow)$
- (4)  $(\Gamma, \bar{\phi} \Rightarrow \Delta)$
- (5)  $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where  $\Gamma$  and  $\Delta$  are meta-multiset variables and in (2) – (5) the variables in any pair of elements in  $\bar{\alpha}$  or  $\bar{\beta}$  or  $\bar{\phi}$  are equal.

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#### Example

$$(\Rightarrow 1)$$
 ,  $(0\Rightarrow)$  ,  $(\Gamma\Rightarrow \top)$  ,  $(\Gamma, \bot\Rightarrow \Delta)$ 

## Example

$$\begin{array}{ccc} \neg 1 \Rightarrow & , & \Rightarrow \neg 0 \\ \phi, \neg \phi \Rightarrow & , & \Rightarrow \phi, \neg \phi \\ \Gamma, \neg \top \Rightarrow \Delta & , & \Gamma \Rightarrow \Delta, \neg \bot \end{array}$$

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## Example

$$\neg 1 \Rightarrow$$
 ,  $\Rightarrow \neg 0$ 

$$\phi, \neg \phi \Rightarrow$$
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$$\Gamma, 
eg op op \Delta$$
 ,  $\Gamma \Rightarrow \Delta, 
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#### Example

The initial sequent  $\Gamma, p, \neg p \land q \Rightarrow \Delta$  is not focussed as the variables of p and  $\neg p \land q$  are not equal.

#### Theorem

- (i) If FL<sub>e</sub> ⊆ L, and L has a (terminating) single-conclusion sequent calculus consisting of semi-analytic rules and focused axioms, then L has Craig (uniform) interpolation.
- (ii) If CFL<sub>e</sub> ⊆ L, and L has a (terminating) multi-conclusion sequent calculus consisting of semi-analytic rules and focused axioms, then L has Craig (uniform) interpolation.

As a positive application we have the following:

### Corollary

The logics  $FL_e$ ,  $FL_{ew}$ ,  $CFL_e$ ,  $CFL_{ew}$ , CPC, and their E, M, MC, EN, MN, K and KD modal versions have the uniform interpolation property.

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As the more interesting negative application we have:

### Corollary

None of the following logics can have a nice proof system:

- Many substructural logics  $(\underline{k}_n, \underline{k}_\infty, R, BL, \cdots)$ ;
- Almost all super-intuitionistic logics (except at most seven of them);
- Almost all extensions of S4 (except at most thirty seven of them);
- The non-normal modal logics **EC** and **ENC**.

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#### Nice Systems (informal)

A system is called **nice** if any provable formula  $\phi$  has an "analytic" proof, i.e., a proof that is only "based on" the *subformulas* of  $\phi$ .

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Usually, having an analytic Hilbert-style proof is too much to expect. But if we enhance the proofs by some **meta-structures**, we may reach the full analyticity.

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• Sequents use the meta-structure  $\phi_1, \dots, \phi_n \Rightarrow \psi$ . Represented as formulas, we have the class  $S = \{(\bigwedge_{i=1}^n \phi_i) \rightarrow \psi \mid n \in \mathbb{N}\}.$ 

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 $\phi_{11}, \cdots, \phi_{1m_1} \Rightarrow \psi_1 \mid \cdots \mid \phi_{n1}, \cdots, \phi_{nm_n} \Rightarrow \psi_n$ . Represented as formulas, we have the class  $\mathcal{H} = \{\bigvee_{i=1}^n [(\bigwedge_{1=1}^{m_i} \phi_{ij}) \to \psi_i] \mid n \in \mathbb{N}\}.$ 

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### Definition

For a class of formulas  $\mathcal{F}$ , a system consisting of **LJ** and some initial sequents is called  $\mathcal{F}$ -analytic if any provable formula  $\phi$  has a proof where all formulas in the proof is the result of a substitution of a subformula of a formula in  $\mathcal{F}$  by the subformulas of  $\phi$ .

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### An Embedding Theorem

We can rewrite any cut-free hypersequent proof system as an  $\mathcal{H}$ -analytic sequent calculus. Specifically, **LC** has a  $\{(p \rightarrow q) \lor (q \rightarrow p)\}$ -analytic sequent calculus.

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To separate the level of sequents and hypersequents:

#### A Separation Theorem

The logic **LC** has no S-analytic sequent calculus.

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# Thank you for your attention!