On the Existence of Proof Systems

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A Negative Line of Research

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- Proposing a convincing **formalization** of what we mean by nice proof systems,
- Finding an **invariant**, i.e., a property that the logic of a nice proof system enjoys,
- And finally, proving that the property is **rare**, i.e., almost all logics in the family do not enjoy the property.

Our Setting

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- Then, we connect the form of the rules to some variants of **interpolation** property of the logic that the system captures.
- As interpolation is a **rare** property, we prove that nice proof systems are rare.

Semi-analytic rules

- *Left semi-analytic rule*:

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

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Example

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta} \qquad \frac{\Gamma_1, \phi \Rightarrow \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \phi + \psi \Rightarrow \Delta_1, \Delta_2}$$

$$\frac{\Pi \Rightarrow \phi \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \Pi, \phi \to \psi \Rightarrow \Delta}$$

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More Concrete Examples

Example

The following rules are semi-analytic:

- The usual conjunction, disjunction and implication rules for **IPC**;
- All the rules in sub-structural logic $\mathbf{FL_e}$, weakening and contraction rules;
- The following rules for exponentials in linear logic:

$$\frac{\Gamma, !\phi, !\phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

Some Non-examples

Example

- The cut rule; since it does not meet the variable occurrence condition.
- the following rule in the calculus of **KC**:

$$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}$$

  in which Δ should consist of negation formulas is not a multi-conclusion semi-analytic rule, simply because the context is not free for all possible substitutions.

Focused axioms

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A sequent is called a *focused axiom* if it has the following form:

(1) $(\phi \Rightarrow \phi)$

(2) $(\Rightarrow \bar{\alpha})$

(3) $(\bar{\beta} \Rightarrow)$

(4) $(\Gamma, \bar{\phi} \Rightarrow \Delta)$

(5) $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where Γ and Δ are meta-multiset variables and in $(2) - (5)$ the variables in any pair of elements in $\bar{\alpha}$ or $\bar{\beta}$ or $\bar{\phi}$ are equal.

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Example

$$(\Rightarrow 1) \quad , \quad (0 \Rightarrow) \quad , \quad (\Gamma \Rightarrow \top) \quad , \quad (\Gamma, \bot \Rightarrow \Delta)$$

More Examples and Non-examples

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$$\neg 1 \Rightarrow \quad , \quad \Rightarrow \neg 0$$

$$\phi, \neg\phi \Rightarrow \quad , \quad \Rightarrow \phi, \neg\phi$$

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Example

The initial sequent $\Gamma, p, \neg p \wedge q \Rightarrow \Delta$ is not focussed as the variables of p and $\neg p \wedge q$ are not equal.

The Main Result

Theorem

(*i*) *If* $\mathbf{FL_e} \subseteq L$*, and* L *has a (terminating) single-conclusion sequent calculus consisting of semi-analytic rules and focused axioms, then* L *has Craig (uniform) interpolation.*

(*ii*) *If* $\mathbf{CFL_e} \subseteq L$*, and* L *has a (terminating) multi-conclusion sequent calculus consisting of semi-analytic rules and focused axioms, then* L *has Craig (uniform) interpolation.*

Applications

As a positive application we have the following:

Corollary

The logics $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, **CPC**, *and their* **E**, **M**, **MC**, **EN**, **MN**, **K** *and* **KD** *modal versions have the uniform interpolation property.*

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As the more interesting negative application we have:

Corollary

None of the following logics can have a **nice** *proof system:*

- *Many substructural logics* ($Ł_n$, $Ł_\infty$, R, BL, $\cdots$)*;*
- *Almost all super-intuitionistic logics (except at most seven of them);*
- *Almost all extensions of* **S4** *(except at most thirty seven of them);*
- *The non-normal modal logics* **EC** *and* **ENC**.

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Nice Systems (informal)

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Usually, having an analytic Hilbert-style proof is too much to expect. But if we enhance the proofs by some **meta-structures**, we may reach the full analyticity.

Niceness as Boundedness

- **Sequents** use the meta-structure $\phi_1, \cdots, \phi_n \Rightarrow \psi$. Represented as formulas, we have the class $\mathcal{S} = \{(\bigwedge_{i=1}^{n} \phi_i) \to \psi \mid n \in \mathbb{N}\}$.

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- **Hypersequents** use the meta-structure $\phi_{11}, \cdots, \phi_{1m_1} \Rightarrow \psi_1 \mid \cdots \mid \phi_{n1}, \cdots, \phi_{nm_n} \Rightarrow \psi_n$. Represented as formulas, we have the class $\mathcal{H} = \{\bigvee_{i=1}^{n}[(\bigwedge_{1=1}^{m_i} \phi_{ij}) \to \psi_i] \mid n \in \mathbb{N}\}$.

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- In a nice sequent-style system, a provable formula ϕ has a proof not consisting of subformulas of ϕ, but by the formulas in the form $(\bigwedge_{i=1}^{n} \phi_i) \to \psi$, where ϕ_i and ψ are *subformulas* of ϕ.

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Definition

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To separate the level of sequents and hypersequents:

A Separation Theorem

The logic **LC** has no $\mathcal{S}$-analytic sequent calculus.

Thank you for your attention!