Feasible Visser-Harrop Property for Intuitionistic Modal Logics

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- **Introducing an essential disjunction:**

$$\frac{\Gamma \Rightarrow \neg(p \wedge q)}{\Gamma \Rightarrow \neg p \vee \neg q} \qquad \frac{\Gamma, p, q \Rightarrow \bot \qquad \Gamma, \neg p \Rightarrow \Delta \qquad \Gamma, \neg q \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

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- **Eliminating a nested implication:**

$$\frac{}{\Gamma, \neg\neg p \Rightarrow p} \qquad \frac{\Gamma \Rightarrow \neg\neg p}{\Gamma \Rightarrow p} \qquad \frac{\Gamma, \neg p \Rightarrow \bot}{\Gamma \Rightarrow p}$$

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If we define a **nice rule** as a rule avoiding the previous two problematic types, we reach the rules that transform formulas **without nested implications** to formulas **without essential disjunctions**. Then:

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- Cut is nice in this sense and hence we can also address the calculi with **explicit cuts**.
- Even proving the special case of feasible disjunction property ($I = \emptyset$) for **LJ** $+$ *Cut* is a highly non-trivial result. [Buss, Mints], [Buss, Pudlák], [Ferrari, et al].

Intuitionistic Modal Logics

Set $\mathcal{L}$ as the propositional language augmented by the two modalities $\Box$ and $\Diamond$. Define the basic intuitionistic modal logic i**K** as the logic of the sequent system **LJ** (including cut) plus the following rules:

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ (K_{\Box}) \qquad \frac{\Gamma, A \Rightarrow B}{\Box\Gamma, \Diamond A \Rightarrow \Diamond B}\ (K_{\Diamond})$$

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It is possible to add any of the following rules to the system $i\mathbf{K}$ to reach different systems for different intuitionistic modal logics.

Some Modal Rules

$$\frac{\Gamma \Rightarrow \Diamond\bot, \Delta}{\Gamma \Rightarrow \bot, \Delta}\ (\Diamond\bot) \qquad \frac{\Gamma \Rightarrow \Diamond(p \vee q), \Delta}{\Gamma \Rightarrow \Diamond p, \Diamond q, \Delta}\ (\Diamond\vee) \qquad \frac{\Gamma, \Diamond p \Rightarrow \Box q}{\Gamma \Rightarrow \Box(p \to q)}\ (\Box\to)$$

$$\frac{}{\Gamma, \Box\bot \Rightarrow \Delta}\ (D_a) \qquad \frac{}{\Gamma \Rightarrow \Diamond\top, \Delta}\ (D_b) \qquad \frac{\Gamma \Rightarrow \Box p, \Delta}{\Gamma \Rightarrow \Diamond p, \Delta}\ (D)$$

$$\frac{\Gamma \Rightarrow \Box p, \Delta}{\Gamma \Rightarrow p, \Delta}\ (T_a) \qquad \frac{\Gamma \Rightarrow p, \Delta}{\Gamma \Rightarrow \Diamond p, \Delta}\ (T_b) \qquad \frac{\Gamma \Rightarrow \Box p, \Delta}{\Gamma \Rightarrow \Box\Box p, \Delta}\ (4_a) \qquad \frac{\Gamma \Rightarrow \Diamond\Diamond p, \Delta}{\Gamma \Rightarrow \Diamond p, \Delta}\ (4_b)$$

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$$\frac{\Gamma \Rightarrow \Box^n p, \Delta}{\Gamma \Rightarrow \Box^m p, \Delta}\ (4_a^{n,m}) \qquad \frac{\Gamma \Rightarrow \Diamond^m p, \Delta}{\Gamma \Rightarrow \Diamond^n p, \Delta}\ (4_b^{n,m})$$

$$\frac{\{\Gamma \Rightarrow \Box^i p, \Delta\}_{i=0}^n}{\Gamma \Rightarrow \Box^{n+1} p, \Delta}\ (tra_a^n) \qquad \frac{\Gamma \Rightarrow \Diamond^{n+1} p, \Delta}{\Gamma \Rightarrow \{\Diamond^i p\}_{0 \leqslant i \leqslant n}, \Delta}\ (tra_b^n)$$

$$\frac{\Gamma \Rightarrow \Diamond\Box p, \Delta}{\Gamma \Rightarrow \Box\Diamond p, \Delta}\ (ga) \qquad \frac{\Gamma \Rightarrow \Diamond^k \Box^l p, \Delta}{\Gamma \Rightarrow \Box^m \Diamond^n p, \Delta}\ (ga_{klmn})$$

$$\frac{\{\Gamma \Rightarrow \Diamond p_i, \Delta\}_{i=0}^n}{\Gamma \Rightarrow \{\Diamond(p_i \wedge (p_j \vee \Diamond p_j))\}_{i \neq j}, \Delta}\ (BW_{n,a}) \qquad \frac{\{\Gamma \Rightarrow \Box(p_i \vee (p_j \wedge \Box p_j)), \Delta\}_{i \neq j}}{\Gamma \Rightarrow \{\Diamond p_i\}_{i=0}^n, \Delta}\ (BW_{n,b})$$

$$\frac{\Gamma \Rightarrow bd_n^a, \Delta}{\Gamma \Rightarrow p_{n+1}, \Delta}\ (BD_{n,a}) \qquad \frac{\Gamma \Rightarrow p, \Delta}{\Gamma \Rightarrow \Box(\Diamond p \to p), \Delta}\ (H_a) \qquad \frac{\Gamma \Rightarrow \Diamond(\Box p \wedge q), \Delta}{\Gamma \Rightarrow \Box(\Diamond p \vee q), \Delta}\ (dir)$$

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The formula bd_n^a is defined recursively: $bd_1^a = \Diamond\Box p_1$ and $bd_{n+1}^a = \Diamond(\Box p_{n+1} \wedge bd_n^a \wedge \neg p_n)$.

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Example

	Is	**Isn't**
Basic	$(p \wedge q)$, $(p \vee q)$, $\Diamond^n p$	$\neg p$, $\Box p$
Almost positive	$\neg p$, $\Diamond^k \Box^l p$	$\neg\neg p$
Almost negative	$\Box^m \Diamond^n p$	$(p \vee \neg p)$, $\Diamond\Box p$

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Note that p, $(p \wedge q)$, $(p \vee q)$, $(p \to q)$, $\neg p$, $\Box p$ and $\Diamond p$ are both almost positive and almost negative.

Almost Negative Rules as the Nice Rules

- I is a finite index set (possibly empty), Γ and Δ multiset variables,
- $\overline{M}$ and $\overline{M'_i}$ multisets of almost positive formulas, and
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- *Left almost negative*:

$$\frac{\{\Gamma, \overline{N'_i} \Rightarrow \overline{M'_i}, \Delta\}_{i \in I}}{\Gamma, \overline{M} \Rightarrow \Delta}$$

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- *Right almost negative*:

$$\frac{\{\Gamma, \overline{N'_i} \Rightarrow \overline{M'_i}\}_{i \in I}}{\Gamma \Rightarrow \overline{N}} \text{ (context-free)} \qquad \frac{\{\Gamma \Rightarrow \overline{M'_i}, \Delta\}_{i \in I}}{\Gamma \Rightarrow \overline{N}, \Delta} \text{ (contextual)}$$

$\overline{N'_i}$ consists of basic formulas, for any $i \in I$. Moreover, if $\overline{N}$ has more than one formula, then all of them must be basic.

Example

All the rules of **LJ** (including cut) and all the rules we had before are almost negative. For instance:

$$\frac{\Gamma \Rightarrow \Diamond^k \Box^l p, \Delta}{\Gamma \Rightarrow \Box^m \Diamond^n p, \Delta} \; (ga_{klmn})$$

is almost negative and covers all the following modal rules:

$$\frac{\Gamma \Rightarrow \Box p, \Delta}{\Gamma \Rightarrow p, \Delta} \; (T_a) \qquad \frac{\Gamma \Rightarrow p, \Delta}{\Gamma \Rightarrow \Diamond p, \Delta} \; (T_b) \qquad \frac{\Gamma \Rightarrow \Box p, \Delta}{\Gamma \Rightarrow \Box\Box p, \Delta} \; (\mathbf{4}_a) \qquad \frac{\Gamma \Rightarrow \Diamond\Diamond p, \Delta}{\Gamma \Rightarrow \Diamond p, \Delta} \; (\mathbf{4}_b)$$

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Some Definitions

Definition

A calculus G (logic L) is called *T-free* if it extends $i\mathbf{K}$ and is valid in the irreflexive Kripke frame of one node. It is called *T-full* if it is valid in the reflexive Kripke frame of one node and extends $i\mathbf{K} + T_a + T_b$.

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Any system consisting of $i\mathsf{K}$ together with any combination of the rules $\{\Diamond\bot, \Diamond\vee, \Box\rightarrow, \{T_a, T_b\}, 4, 4^{n,m}, tr^n, 5, B, BD_{n,a}, BW_{n,a}, H_a, ga, dir\}$ is either T-free or T-full. The system $i\mathsf{K} + D$ is neither T-free nor T-full.

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Definition

The set of Harrop formulas is the smallest set of $\mathcal{L}$-formulas including atomic formulas, $\bot, \top$, and is closed under $\wedge, \Box$, and implications of the form $A \rightarrow B$, where A is an arbitrary formula and B is a Harrop formula.

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Let G be a T-free or a T-full calculus extending $i\mathbf{K}$ only by some almost negative rules. Then G feasibly admits the Visser-Harrop rules, i.e., for any G-proof π of a sequent $\Gamma, \{A_i \to B_i\}_{i \in I} \Rightarrow C \vee D$ in G, where Γ is a set of Harrop formulas, there is a G-proof either for $\Gamma, \{A_i \to B_i\}_{i \in I} \Rightarrow C$ or $\Gamma, \{A_i \to B_i\}_{i \in I} \Rightarrow D$ or $\Gamma, \{A_i \to B_i\}_{i \in I} \Rightarrow A_i$, for some $i \in I$ and the algorithm to find the proof is polynomial time in π.

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Applications

- Any system extending $i\mathbf{K}$ only by a combination of the rules $\{\Diamond\bot, \Diamond\vee, \Box\to, \{T_a, T_b\}, 4, 4^{n,m}, tr^n, 5, B, BD_{n,a}, BW_{n,a}, H_a, ga, dir\}$ feasibly admits the Visser-Harrop rules.

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- If a T-free or a T-full logic does not admit the Visser rules (e.g. any extension of **LC**), then it has no calculus extending $i\mathbf{K}$ only by the almost negative rules.

Thank you for your attention!