Generalized Heyting Algebras and Duality

Amirhossein Akbar Tabatabai*, Majid Alizadeh,
Masoud Memarzadeh

Abstract

∇-algebras are the natural generalization of Heyting algebras, unifying many algebraic structures including bounded lattices, Heyting algebras, temporal Heyting algebras and point-free formalization for dynamical systems. They are also powerful enough to represent any "natural" abstract modality and any abstract implication.
In this paper, we will study the algebraic and topological properties of different varieties of these algebras. These investigations starts with the algebraic theme of characterizing the sub-directly and simple ∇-algebras, the Dedekind-MacNeille completion, the canonical construction and the amalgamation property. Then, we will move to the topological theme to provide a unification of Priestley and Esakia dualities to capture the dual spaces corresponding to these algebras. This, then leads to spectral duality and finally to some ring-theoretic representation for some of these algebras.

1 Introduction

Implication is the logical constant developed to internalize the meta-level provability order between the propositions inside the object language itself. As this provability order has many different formalizations and in each formalization there are many different structures to internalize, the variety of implications covers a wide-range family of implications from the classical and intutionistic implications to weak sub-intuitionistic and sub-structural ones. Following this internalization philosophy, the first author introduced a general notion of implication to unify these different implications [1]. Here, we restrict ourselves to a special case of that definition that suits our interest the best. To start, let $(A, \leq, \wedge, 1)$ be a bounded meet-semilattice

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formalizing the world of propositions augmented with the provability order $\leq$, the conjunction $\wedge$ and the value true 1. An implication $\to$ is a binary operator internalizing the fact that the order $\leq$ is reflexive and transitive. Formally, an implication $\to$ over $(A, \leq, \wedge, 1)$ is an order-preserving operator from $(A, \leq)^{op} \times (A, \leq)$ to $(A, \leq)$ such that:

- (internal reflexivity) $a \to a = 1$,
- (internal transitivity) $(a \to b) \wedge (b \to c) \leq a \to c$.

and it is called meet-internalizing, if $a \to b \wedge c = (a \to b) \wedge (a \to c)$. Over any bounded meet-semilattice there are many possible implications, from the simplest for which $a \to b = 1$, for any $a, b \in A$ to Heyting implication, if it exits. Comparing this general definition to the specific case of Heyting implication, it is clear that the abstract implications have many instances at the cost of breaking the adjunction that the Heyting implication presents. As the adjunction means that we have full introduction-elimination rules for implication, the lack of such adjunction means that the full understanding of all implication constants is somewhat missing. To solve this problem, [1] also introduces a generalization of Heyting implication to provide a well-behaved adjunction-style form of implication that is powerful enough to represent all abstract implications. In this sense, these generalized Heyting implications can be considered as the natural and general formalization for implications as internalizers. Formally, a ∇-algebra is a tuple $(A, \leq, \wedge, \vee, 0, 1, \nabla, \to)$, where $(A, \leq, \wedge, \vee, 0, 1)$ is a bounded lattice, $\nabla : A \to A$ is a unary operation and $\to : A \times A \to A$ is a binary operation such that $\nabla(-) \wedge a \dashv a \to (-)$, for any $a \in A$. A ∇-algebra is called normal if ∇ commutes with all finite meets. ∇-algebras are the clear generalization of both Heyting algebras ($\nabla = id$) and bounded lattices ($\nabla a = 0$ and $a \to b = 1$). As we have already mentioned, ∇-algebras are powerful enough to represent all implications:

Theorem 1.1. *(Representation Theorem) Let $\mathcal{A}$ be a bounded meet-semilattice and $\to$ be an implication on $\mathcal{A}$. Then, there exists a locale $\mathcal{X}$, a ∇-algebra $(\mathcal{X}, \nabla, \to_{\mathcal{X}})$, an order-preserving map $F : \mathcal{X} \to \mathcal{X}$ and a bounded meet semilattice embedding $i : \mathcal{A} \to \mathcal{X}$ such that $i(a \to_{\mathcal{A}} b) = F(i(a)) \to_{\mathcal{X}} F(i(b))$. Moreover, if $\to$ is meet-internalizing, then F can be set as the identity function and hence i also maps the implication of $\mathcal{A}$ to the implication of $\mathcal{X}$.*

Apart from representing all abstract implications, investigating ∇-algebras have two other strong motives. First, they provide an *algebraic model for the basic intuitionistic tense logics* and secondly they can be read as the *point-free dynamical systems*. We will explain these two motives in the following.

The algebraic models for classical tense logic first appeared in [6] as boolean algebras together with two adjunctions. Then, as tense logics with intuitionistic base became interesting [15], the algebraic models updated accordingly [11]. On the other hand, if $\mathcal{A}$ is a Heyting algebra with the Heyting implication $\supset$, then it is easy to see that the implication of a ∇-algebra over $\mathcal{A}$ simplifies to $a \to b = \Box(a \supset b)$, where $\Box a = 1 \to a$. This means that a Heyting ∇-algebra is nothing but a Heyting algebra together with an adjunction $\nabla \dashv \Box$ which clearly is the algebraic model for the very basic intuitionistic tense logic. Motivated by this situations, we will provide an order-theoretic representation for many varieties of ∇-algebras.

Dynamic topological systems are defined as th pairs (X, f), where X is a topological space encoding the space of states and $f : X \to X$ is a continuous map encoding the dynamism of the system [2]. To address the universal properties of these systems, [26] introduced some logical calculi to capture the behavior of the systems. Later, in quest of finding a decidable fragment of these systems and following an unpublished suggestion of Kamide, Fernandez Douque studied the intuitionistic version of dynamical topological logic [16]. Towards investigating the intuitionistic version of dynamical systems and following the more constructive reading of spaces as locales, it is reasonable to introduce the point-free version of dynamical spaces. In the point-free world, a topological space must be replaced with a locale and a continuous function is nothing but a localic map $f^* : \mathcal{X} \to \mathcal{X}$ preserving all joins and all finite meets. The definition is clearly non-elementary. To make it more amenable for elementary logical investigations it is reasonable to use an equivalent definition using adjunctions: a continuous map is a pair of $f^* \dashv f_*$ where f^* commutes with all finite meets. In this sense, a point-free version of a dynamical space must be the tuple $(\mathcal{X}, f^*, f_*)$, where $f^* \dashv f_*$ and f^* preserves all finite meets. These structures are exactly what we get from normal ∇-algebras over $\mathcal{A}$, where $\mathcal{A}$ is a locale. The adjunction $\nabla \dashv \Box$ captures a point-free version of a continuous function and hence the whole ∇-algebra is a point-free version of a dynamical space (X, f), where $f : X \to X$ is a continuous map. There are some topological properties that are even representable at this level. For instance, it is well-known that under some separation conditions on the space X, the map f is an embedding (surjective) if f^* is surjective (one-to-one). Motivated by these situations, we will investigate ∇-algebras with surjective and injective ∇'s that we call faithful and full ∇-algebras, respectively.

Inspired by the three aforementioned aspects of ∇-algebras, in this paper, we will provide an extensive algebraic and topological study of different varieties of distributive, Heyting, normal, faithful and full ∇ algebras. We aim to study the structure of different varieties of ∇-algebras by providing a characterization for sub-directly irreducible and simple ∇-algebras and we will show how complex a simple finite ∇-algebra can be. Then, we will move to the completions of ∇-algebras, Dedekind-MacNeille completion and the usual canonical construction. The latter Kripke-style representation for distributive ∇-algebras provides an interesting representation for normal ∇-algebra as dynamic posets. It also helps to prove the amalgamation property for some varieties of distributive normal ∇-algebras and hence deductive interpolation for basic logics of point-free dynamical systems. Finally, we unify Priestley and Esakia dualities to provide the duality theory for distributive ∇-algebras that leads to the corresponding spectral duality à la [7]. We will use the spectral duality for normal ∇-algebras to provide a ring-theoretic representation that interprets any normal ∇-algebra as a dynamical ring.

The structure of the paper is as follows. In Section 2, we will cover the preliminary notions to be able to introduce the varieties of ∇-algebras in Section 3. Then, in Section 4, we aim to study the structure of different varieties of ∇-algebras by providing a characterization for sub-directly irreducible and simple ∇-algebras. Then, we move to the completions of ∇-algebras. Section 5 is devoted to Dedekind-MacNielle completion while in Section 6, we investigate the usual canonical construction and Kripke-style representation of the algebras. This helps to read some ∇-algebras as the models for intuitionistic tense logics and some others as the point-free dynamical systems. It also helps to prove the amalgamation property for some varieties of ∇-algebras. In Section 7, we introduce the logical systems for the new implication to provide a natural syntax for the algebraic, topological and Kripke-style structures. In Section 8, we unify Priestley and Esakia dualities to provide the duality theory of ∇-algebras and then the corresponding spectral duality à la [7]. Finally, in Section 9, we use the spectral duality to provide some ring-theoretic representation for some families of ∇-algebras.

2 Preliminaries

In this section, we will present the required preliminaries. In the algebraic part that is our main emphasis, we will be quite extensive. In topological and categorical parts, though, we only recall some useful points.

A poset $\mathcal{P} = (P, \leq)$ is a pair of a set and a reflexive, anti-symmetric and transitive binary relation $\leq \subseteq P^2$. By $\mathcal{P}^{op}$, we mean the poset $(P, \geq)$, namely P with the opposite order. A subset $S \subseteq P$ is called a downset, if it is downward closed, i.e., for every $x \in S$ and any $y \in P$, if $y \leq x$ we have $y \in S$. An upset is defined dually. For any subset $S \subseteq P$, by its downset, denoted by $\downarrow S$, we mean the least downset extending S and by the upset of S, denoted by $\uparrow S$, we mean the least upset extending S. When $S = \{a\}$, sometimes we denote $\downarrow a$ by $(a]$ and $\uparrow a$ by $[a)$. The set of all upsets and downsets of $(P, \leq)$ are denoted by $U(P, \leq)$ and $D(P, \leq)$, respectively.

If $S \subseteq P$. Then, if the greatest lower bound of S exists, it is called the meet of the elements of S and is denoted by $\bigwedge S$. If $S = \{a, b\}$, the meet $\bigwedge S$ is usually denoted by $a \wedge b$. A poset is called complete if for any set $S \subseteq P$, the meet $\bigwedge S$ exists. If the least upper bound of S exists, it is called the join of the elements of S and is denoted by $\bigvee S$. If $S = \{a, b\}$, the join $\bigvee S$ is usually denoted by $a \vee b$. A poset is called a meet semi-lattice, if for any $a, b \in P$, the meet $a \wedge b$ exists. It is called bounded, if it also has a greatest element, denoted by 1. A meet semi-lattice is called a lattice, if for any $a, b \in P$, the join $a \vee b$ also exits. A lattice is called bounded, if it has both the greatest and the least elements, denoted by 1 and 0, respectively. A bounded lattice is called distributive, if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

for any $a, b, c \in P$. A bounded lattice is called a locale, if for any $S \subseteq P$, the join $\bigvee S$ exists and

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i),$$

A subset of a bounded lattice is called filter, if it is an upset and closed under the finite meets. The filters in the form $[a)$ are called the principal filters. The set of all filters of the lattice $\mathcal{A}$ is denoted by $\mathcal{F}(\mathcal{A})$. A filter is called prime if $0 \notin P$ and $a \vee b \in P$ implies either $a \in P$ or $b \in P$. The set of all prime filters of a lattice $\mathcal{A}$ is denoted by $\mathcal{F}_p(\mathcal{A})$. A subset of $\mathcal{A}$ is called an ideal, if it is a downset and closed under the finite joins. The ideals in the form $(a]$ are called the principal ideals. The set of all ideals of a lattice $\mathcal{A}$ is denoted by $\mathcal{I}(\mathcal{A})$.

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets and $f : P \to Q$ be a function. It is called a poset map, if it preserves the order, meaning $f(p) \leq_Q f(q)$ for any $p \leq_P q$. An order-preserving map is called an order embedding, if for any $p, q \in P$, the inequality $f(p) \leq_Q f(q)$ implies $p \leq_P q$. A poset map between two meet-semilatices is called a meet-semilatic map, if it preserves the binary meets. Similarly, a poset map is called a meet-semilatic map, if it also preserves the greatest element, it is a lattice map, if it preserves both meets and joins and it is a locale map if it preserves all finite meets and all possible joins.

Definition 2.1. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets and $f : P \to Q$ and $g : Q \to P$ be two order-preserving maps. The function f is called the left adjoint for g (and g is called the right adjoint for f) denoted by $f \dashv g$ iff

$$f(a) \leq_Q b \quad \text{iff} \quad a \leq_P g(b)$$

for any $a \in P$ and $b \in Q$. In such situation the pair (f, g) is called an adjunction.

Theorem 2.2. *[8] Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets and $f : P \to Q$ and $g : Q \to P$ be two order-preserving maps such that $f \dashv g$. Then:*

- *The following inequalities hold: $fg(q) \leq_Q q$ and $p \leq_P gf(p)$, for any $p \in P$ and $q \in Q$.*
- *The following two equalities hold: $fgf(p) = f(p)$ and $gfg(q) = g(q)$, for any $p \in P$ and $q \in Q$.*
- *f is one-to-one iff g is surjective iff $gf(p) = p$, for any $p \in P$.*
- *f is surjective iff g is one-to-one iff $fg(q) = q$, for any $q \in Q$.*

Theorem 2.3. *(Adjoint functor theorem for posets) Let $(P, \leq_P)$ be a complete poset and $\mathcal{Q} = (Q, \leq_Q)$ be a poset. Then an order preserving map $f : (P, \leq_P) \to (Q, \leq_Q)$ has a right (left) adjoint iff it preserves all joins (meets).*

Proof. See [8]. □

Theorem 2.4. *[14](Prime filter theorem) Let $\mathcal{A}$ be a distributive lattice, F a filter and I an ideal such that $F \cap I = \emptyset$. Then there exists a prime filter P such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Let X be a topological space. We denote its locale of open subsets by $\mathcal{O}(X)$. A topological space is called T_0, if for any two different points $x, y \in X$, there is an open set which contains one of these points and not the other. It is called T_D, if for any $x \in X$, there is an open U such that $x \in U$ and $U - \{x\}$ is open, as well. A space is called sober if for every closed set C that is not the union of two smaller closed sets, there is a unique point $x \in X$ such that $C = Cl(\{x\})$. A space is Hausdorff if for any $x, y \in X$ if $x \neq y$, then there are two opens $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$. A continuous map is called a topological embedding if f induces a homeomorphism between X and $f[X]$. By the specialization pre-order on a space X, we mean the binary relation $\leq \subseteq X \times X$ defined as $x \leq y$ iff for any open U, if $x \in U$ then $y \in U$. It is a partial order if the space is T_0. It is clear that if $f : X \to Y$ is a continuous map and $\leq_X$ and $\leq_Y$ are the specialization pre-orders of X and Y, respectively, then f is order preserving.

Theorem 2.5. *[25] Let X and Y be two topological spaces and $f : X \to Y$ be a continuous map. Then,*

- *If f is surjective, then $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is one-to-one. The converse is true, if Y is T_D.*
- *If f is a topological embedding, then $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective. The converse is also true, if X is T_0.*

3 ∇-Algebras

Following the idea that implications are the internalizations of the order relation of the base poset, [1] introduced a general formalization for implications as the binary operators that at least internalize the fact that the order is reflexive and transitive. In its special form, it means:

Definition 3.1. Let $\mathcal{A} = (A, \leq, \wedge, 1)$ be a bounded meet semi-lattice. A binary order-preserving operator $\to$ from $\mathcal{A}^{op} \times \mathcal{A}$ to $\mathcal{A}$ is called an implication if for any $a, b, c \in A$:

- (*internal reflexivity*) $a \to a = 1$,
- (*internal transitivity*) $(a \to b) \wedge (b \to c) \leq a \to c$.

An implication is called *meet-internalizing* if $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$ and *join-internalizing* if $\mathcal{A}$ is also a bounded lattice and $(a \vee b) \to c = (a \to c) \wedge (b \to c)$, for any $a, b, c \in A$. The tuple $(A, \leq, \wedge, 1, \to)$ is called a *strong algebra* if $\to$ is an implication over $(A, \leq, \wedge, 1)$ and it is called meet-internalizing (join-internalizing) strong algebra if its implication is meet-internalizing (join-internalizing). In any strong algebra, $\Box a$ is an abbreviation for $1 \to a$. If $\mathcal{A} = (A, \leq_A, \wedge_A, 1_A)$ and $\mathcal{B} = (B, \leq_B, \wedge_B, 1_B)$ are two strong algebras, by a *strong algebra morphism*, we mean a map $f : A \to B$, preserving the order, all the finite meets and the implication. It is called an *embedding* if it is also an order embedding.

As explained in [1], one of the main sources of implication is the spatio-temporal setting of a locale augmented with a temporal-like join-preserving operator. To be more precise, let $\mathcal{X}$ be a locale and $\nabla : \mathcal{X} \to \mathcal{X}$ be a join preserving function. Define the implication as $a \to b = \bigvee\{c \in \mathcal{X} \mid \nabla c \wedge a \leq b\}$. It is not hard to see that $(\mathcal{X}, \to)$ is a meet- and join-internalizing strong algebra, see [1] or Theorem 3.4 below. The main ingredient of the proof is the fact that the operation $\nabla(-) \wedge a$ preserves all joins, for any element a. This property addresses all the arbitrary subsets of $\mathcal{X}$ which is non-elementary in nature. To make the structure elementary, it is reasonable to replace the join-preserving *property* with a *structure*, namely the right adjoint of the operator $\nabla(-) \wedge a$. The situation is similar to that of Heyting algebras as the elementary version of the locales, replacing the distributivity of meets over the arbitrary joins with the Heyting implication. This leads to the following definition, first appeared in [1]:

Definition 3.2. Let $\mathcal{A} = (A, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A tuple $(\mathcal{A}, \nabla, \to)$ is called a ∇-algebra if $\nabla c \wedge a \leq b$ is equivalent to $c \leq a \to b$, for any $a, b, c \in A$, or in a more abstract term $\nabla(-) \wedge a \dashv a \to (-)$, for any $a \in A$.

Remark 3.3. Here are two remarks. First, note that we have the useful adjunction $\nabla \dashv \Box$, by substituting $a = 1$ in the definition of a ∇-algebra. Secondly, notice that the adjunction in the definition of ∇-algebras implies that both ∇ and $\to$ are order-preserving. For ∇, if $c \leq d$, for any $b \in A$, we have

$$\nabla d \leq b \quad \text{then} \quad d \leq 1 \to b \quad \text{then} \quad c \leq 1 \to b \quad \text{then} \quad \nabla c \leq b.$$

Now, set $b = \nabla d$ to prove $\nabla c \leq \nabla d$. A similar proof works for $\to$.

The following theorem ensures that any such adjunction leads to an implication, as we have claimed before. This theorem, as easy as it is, was originally proved in [1]. However, we present its proof again in order to make the present paper as self-contained as possible.

Theorem 3.4. *If $(\mathcal{A}, \nabla, \to)$ is a ∇-algebra, then the structure $(\mathcal{A}, \to)$ is a meet-internalizing strong algebra. If $\mathcal{A}$ is also distributive, the strong algebra will be join-preserving, as well.*

Proof. Since $\nabla 1 \wedge a \leq a$ we have $1 \leq a \to a$ and hence $a \to a = 1$. Secondly,

$$[d \leq (a \to b) \wedge (b \to c)] \quad \Rightarrow \quad [d \leq (a \to b) \quad \text{and} \quad d \leq (b \to c)] \quad \Rightarrow$$

$$[\nabla d \wedge a \leq b \quad \text{and} \quad \nabla d \wedge b \leq c] \quad \Rightarrow \quad [\nabla d \wedge a \leq c] \quad \Rightarrow \quad [d \leq a \to c].$$

Hence, $(a \to b) \wedge (b \to c) \leq a \to c$. For the meet-internalizing condition, we have

$$[d \leq a \to b \wedge c] \quad \text{iff} \quad [\nabla d \wedge a \leq b \wedge c] \quad \text{iff} \quad [\nabla d \wedge a \leq b \quad \text{and} \quad \nabla d \wedge a \leq c] \quad \text{iff}$$

$$[d \leq a \to b \quad \text{and} \quad d \leq a \to c] \quad \text{iff} \quad [d \leq (a \to b) \wedge (a \to c)].$$

Therefore, $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$. The proof for the join-internalizing condition is similar, but uses distributivity. □

The following representation theorem shows that the implications arised from these adjunctions are genral enough to represent all implications. A weaker version of the theorem first proved in the unpublished draft of the present paper and then generalized in [1] to its current form:

Theorem 3.5. *(Representation Theorem [1]) Let $\mathcal{A}$ be a strong algebra. Then, there exists a locale $\mathcal{X}$, a ∇-algebra $(\mathcal{X}, \nabla, \to_{\mathcal{X}})$, an order-preserving map $F : \mathcal{X} \to \mathcal{X}$ and a bounded meet semi-lattice embedding $i : \mathcal{A} \to \mathcal{X}$ such that $i(a \to_{\mathcal{A}} b) = F(i(a)) \to_{\mathcal{X}} F(i(b))$. Moreover, if $\mathcal{A}$ is meet-internalizing, then F can be set as the identity function and hence i will be a strong algebra morphism. If $\mathcal{A}$ is also distributive and join-internalizing, then i can be set as a bounded lattice morphism.*

This representation theorem makes ∇-algebras interesting, as they present the full adjunction situation on the one hand and capture all possible implications, on the other.

Definition 3.6. Let $(\mathcal{A}, \nabla, \to)$ be a ∇-algebra:

- (D) If $\mathcal{A}$ is distributive, the ∇-algebra is called distributive.
- (H) If $\mathcal{A}$ is a Heyting algebra, the ∇-algebra is called Heyting.
- (N) If ∇ commutes with all finite meets, i.e., $\nabla 1 = 1$ and $\nabla(a \wedge b) = \nabla a \wedge \nabla b$, for any $a, b \in A$, then the ∇-algebra is called normal.
- (R) If $a \leq \nabla a$, for any $a \in A$, the ∇-algebra is called right.
- (L) If $\nabla a \leq a$, for any $a \in A$, the ∇-algebra is called left.
- (Fa) If ∇ is surjective, the ∇-algebra is called faithful.
- (Fu) If $\Box$ is surjective, the ∇-algebra is called full.

The Heyting implication is a structure and not a mere property. Therefore, "Heyting ∇-algebra" is an ambigious notion, as it is not clear that if we include the Heyting implication in the signature of the algebra or not. To solve this ambiguity, when we mean a ∇-algebra that is also Heyting, we call it Heyting ∇-algebra and when we mean an algebra in the form $(\mathcal{A}, \nabla, \to, \supset)$, where $(\mathcal{A}, \nabla, \to)$ is a ∇-algebra and $\supset$ is the Heyting implication over $\mathcal{A}$, we call the structure *explicitly Heyting ∇-algebra*. This difference in the signature is important when we investigate the algebraic or categorical properties of the algebras.

For any $C \subseteq \{D, H, N, R, L, Fa, Fu\}$, by $\mathcal{V}(C)$ we mean the class of all ∇-algebras with the properties described in the set C. For instance, $\mathcal{V}(\{N, D\})$ is the class of all normal distributive ∇-algebras. Sometimes, for simplicity, we omit the brackets. For instance, we write $\mathcal{V}(N, D)$ for $\mathcal{V}(\{N, D\})$ and if $X \in \{D, H, N, R, L, Fa, Fu\}$, we write $\mathcal{V}(X, C)$ for $\mathcal{V}(\{X\} \cup C)$. By $\mathcal{V}_H(C)$ we mean the class of all explicitly Heyting ∇-algebras satisfying the conditions in C. If $(\mathcal{A}, \nabla_A, \to_A)$ and $(\mathcal{B}, \nabla_B, \to_B)$ are two ∇-algebras, by a ∇-algebra morphism, we mean a bounded lattice morphism $f : \mathcal{A} \to \mathcal{B}$ that also preserves ∇ and $\to$ and if both of the ∇-algebras are explicitly Heyting and f also preserves the Heyting implication, it is called a Heyting ∇-algebra morphism. For any $C \subseteq \{D, H, N, R, L, Fa, Fu\}$, the class $\mathcal{V}(C)$ of ∇-algebras together with the ∇-algebra morphisms form a category that we denote by $\mathbf{Alg}_\nabla(C)$. Similarly, the class of $\mathcal{V}_H(C)$ with Heyting ∇-algebra morphisms form a category that we denote by $\mathbf{Alg}^H_\nabla(C)$.

Example 3.7. Any bounded lattice $\mathcal{A}$ with $\nabla a = 0$ and $a \to b = 1$, for any $a, b \in A$ forms a ∇-algebra, because

$$0 = \nabla c \wedge a \leq b \quad \text{iff} \quad c \leq a \to b = 1$$

Moreover, any Heyting algebra with $\nabla a = a$ and $a \to b = a \supset b$, where $\supset$ is the Heyting implication forms a ∇-algebra. In this sense, ∇-algebras are the common generalization of bounded lattices and Heyting algebras.

The following two examples are borrowed from [1] to provide some natural examples for ∇-algebras. We will use them later.

Example 3.8. *(Topological Frames)* Let X be a topological space and $f : X \to X$ be a continuous function. Then, the pair (X, f) is called a *topological frame*. Define $\to_f$ over $\mathcal{O}(X)$ by $U \to_f V = f_*(U^c \cup V)$, where $f_* : \mathcal{O}(X) \to \mathcal{O}(X)$ is the right adjoint of f^{-1}. Note that f_* exists, since f^{-1} preserves the arbitrary unions and $\mathcal{O}(X)$ is complete, by Theorem 2.3. Then, the structure $(\mathcal{O}(X), f^{-1}, \to_f)$ is a normal Heyting ∇-algebra, simply because for any $U, V, W \in \mathcal{O}(X)$, we have

$$f^{-1}(W) \cap U \subseteq V \quad \text{iff} \quad W \subseteq f_*(U^c \cup V) = U \to_f V$$

and f^{-1} commutes with finite intersections. It is also worth mentioning that adding the Heyting implication $\supset$ to the structure results in the explicitly Heyting normal ∇-algebra $(\mathcal{O}(X), f^{-1}, \to_f, \supset)$. Moreover, using Theorem 2.2 and Theorem 2.5, we can observe that if f is surjective, then the algebra $(\mathcal{O}(X), f^{-1}, \to_f)$ is faithful, as surjectivity implies the injectivity of $\nabla = f^{-1}$ and hence the surjectivity of f_*. The converse also holds if X is T_D. In addition, if f is a topological embedding, then the algebra $(\mathcal{O}(X), f^{-1}, \to_f)$ is full, as being an embedding implies the surjectivity of $\nabla = f^{-1}$. The converse also holds if X is T_0.

Example 3.9. *(Kripke Frames)* Let $(W, \leq)$ be a poset. By a *Kripke frame* we mean a tuple $\mathcal{K} = (W, \leq, R)$, where R is a binary relation over W, compatible with the partial order, i.e., if $k' \leq k$, $l \leq l'$ and $(k, l) \in R$ then $(k', l') \in R$, for any $k, k', l, l' \in W$. To any Kripke frame, we can assign a canonical Heyting ∇-algebra, encoding its structure via topology. Set $\mathcal{X}$ as the locale of all upsets of $(W, \leq)$ and define $\nabla : \mathcal{X} \to \mathcal{X}$ as $\nabla_{\mathcal{K}} U = \{x \in W \mid \exists y \in U \; R(y, x)\}$ and $U \to_{\mathcal{K}} V = \{x \in W \mid \forall y \in W[R(x, y) \wedge y \in U \Rightarrow y \in V]\}$. It is easy to see that $(\mathcal{X}, \nabla_{\mathcal{K}}, \to_{\mathcal{K}})$ is a Heyting ∇-algebra and hence $(\mathcal{X}, \nabla_{\mathcal{K}}, \to_{\mathcal{K}}, \supset)$ is an explicitly Heyting ∇-algebra, where $\supset$ is the usual Heyting implication, i.e., $U \supset V = \{x \in W \mid \forall y \in W[(x \leq y) \wedge y \in U \Rightarrow y \in V]\}$

So far, we have seen that the ∇-algebras lead to strong algebras and the latter is powerful enough to represent the former. In the following, we will provide a characterization of implications that comes from ∇-algebras.

Theorem 3.10. *Let $\mathcal{X}$ be a locale and $(\mathcal{X}, \to)$ be a strong algebra. Then, the followings are equivalent:*

- (i) *There is an operator $\nabla : \mathcal{X} \to \mathcal{X}$ such that $(\mathcal{X}, \nabla, \to)$ is a ∇-algebra.*
- (ii) *The operator $\Box$ preserves all arbitrary meets and for any $a, b, c \in \mathcal{X}$, we have $a \to (b \supset c) = a \wedge b \to c$, where $\supset$ is the Heyting implication of $\mathcal{X}$.*
- (iii) *The operator $\Box$ preserves all arbitrary meets and for any $b, c \in \mathcal{X}$, we have $b \to c = \Box(b \supset c)$, where $\supset$ is the Heyting implication of $\mathcal{X}$.*

Proof. To prove (ii) from (i), since $\nabla \dashv \Box$, the operator $\Box$ must preserve all meets. Moreover, we have $x \leq a \to (b \supset c)$ iff $\nabla x \wedge a \leq b \supset c$ iff $\nabla x \wedge a \wedge b \leq c$ iff $x \leq a \wedge b \to c$. Hence, $a \to (b \supset c) = a \wedge b \to c$. To prove (iii) from (ii), just set $a = 1$. Finally, to prove (i) from (iii), since $\Box$ preserves all meets and $\mathcal{X}$ is complete, by the adjoint functor theorem, Theorem 2.3, $\Box$ has a left adjoint. Call it ∇. We claim that this ∇ works, because

$$\nabla x \wedge b \leq c \quad \text{iff} \quad \nabla x \leq b \supset c \quad \text{iff} \quad x \leq \Box(b \supset c) \quad \text{iff} \quad x \leq b \to c.$$

$\Box$

In the following, we will provide an example of a locale and an implication whose $\Box$ preserves all meets, but it can not be a part of a ∇-algebra. Hence, the second conditions in parts (ii) and (iii) of Theorem 3.10 are necessary.

Example 3.11. It is a well-known fact that the inverse image of the continuous functions do not necessarily preserve the Heyting implication of the corresponding locales of the open sets. The counter-example can also be set in a way that the map is defined over one fixed space of upsets of a partial order. We will present such a set up later in this example. But for now, let us show that how this data provides the example we intend to find. By the claim, there exist a topological space X of the upsets of a partial order and a continuous map $f : X \to X$ such that for some open subsets U and V of X, we have $f^{-1}(U \supset V) \neq f^{-1}(U) \supset f^{-1}(V)$, where $\supset$ is the Heyting implication of $\mathcal{O}(X)$. Define $U \to V = f^{-1}(U) \supset f^{-1}(V)$. Since f^{-1} commutes with all arbitrary unions and intersections, it is easy to check that $\to$ is an implication on $\mathcal{X}$. Moreover, since any intersection of the opens is open in any order topology, the meet is actually the intersection and hence the operator $\Box U = 1 \to U = f^{-1}(U)$ is meet-preserving. However, there is no ∇ such that $(\mathcal{X}, \nabla, \to)$ is a ∇-algebra, because if there is, by Theorem 3.10, part (iii), for any opens U and V of X, we must have $U \to V = \Box(U \supset V) = f^{-1}(U \supset V)$, while $U \to V$ is $f^{-1}(U) \supset f^{-1}(V)$. This means $f^{-1}(U \supset V) = f^{-1}(U) \supset f^{-1}(V)$ which is not the case, by the assumption.
To provide the space and the continuous map as used above, set $P = \{a, b, c\}$ and $\leq$ as the partial order generated by the inequalities $a \leq b$ and $a \leq c$. Define $f : P \to P$ as $f(a) = a$ and $f(b) = f(c) = c$ and set X as the topological space of the upsets of $(P, \leq)$. Since f is increasing, it is also continuous. Set $U = \{b\}$ and $V = \emptyset$. Since $f^{-1}(b) = \emptyset$ we have $f^{-1}(b) \supset f^{-1}(\emptyset) = P$, while $(\{b\} \supset \emptyset) = \{c\}$ and hence $f^{-1}(b \supset \emptyset) = \{b, c\}$. Therefore, $f^{-1}(U \supset V) \neq f^{-1}(U) \supset f^{-1}(V)$.

In the rest of this section, we will investigate some of the basic properties of ∇-algebras. We will use these properties to show that for any $C \subseteq \{D, N, R, L, Fa, Fu\}$, the classes $\mathcal{V}(C)$ and $\mathcal{V}_H(C)$ form a variety.

Theorem 3.12. *(Right and Left) Let $\mathcal{A} = (A, \nabla, \to)$ be a ∇-algebra. Then:*

(i) *$\mathcal{A}$ is left iff $a \leq \Box a$, for any $a \in A$.*

(ii) *$\mathcal{A}$ is right iff $(a \to b) \wedge a \leq b$, for any $a, b \in A$ iff $\Box b \leq b$, for any $b \in A$.*

Proof. Part (i) is an easy consequence of the adjunction $\nabla \dashv \Box$. For (ii), if $\mathcal{A}$ is right, then $(a \to b) \wedge a \leq \nabla(a \to b) \wedge a \leq b$, for any $a, b \in A$. The middle condition clearly implies the third condition for $a = 1$ and finally, if $\Box b \leq b$, for any $b \in A$, we have $a \leq \Box \nabla a \leq \nabla a$ which means that the ∇-algebra $\mathcal{A}$ is right. $\Box$

Theorem 3.13. *(Faithfulness) Let $\mathcal{A} = (A, \nabla, \to)$ be a ∇-algebra. Then, the followings are equivalent:*

(i) *$\nabla \Box a = a$, for any $a \in A$.*

(ii) *∇ is surjective.*

(iii) *$a \wedge \nabla(a \to b) = a \wedge b$, for any $a, b \in A$.*

(iv) *$c \to a \leq c \to b$ implies $c \wedge a \leq b$, for any $a, b, c \in A$.*

(v) *$\Box$ is an order embedding, i.e., if $\Box a \leq \Box b$ then $a \leq b$.*

Proof. It is clear that (i) implies (ii). To prove (iii) from (ii), since by the adjunction we have $a \wedge \nabla(a \to b) \leq b$, it is enough to show that $b \leq \nabla(a \to b)$. Since ∇ is surjective, there exists $c \in A$ such that $b = \nabla c$. Since $b \wedge a \leq b$, we have $\nabla c \wedge a \leq b$ which implies $c \leq a \to b$ and hence, $b = \nabla c \leq \nabla(a \to b)$. To prove (iv) from (iii), assume $c \to a \leq c \to b$. Hence, $c \wedge \nabla(c \to a) \leq b$. But by (iii), we have $c \wedge a \leq b$. To reach (v) from (iv), it is just enough to set $c = 1$. And finally, to prove (i) from (v), use Theorem 2.2. $\Box$

Remark 3.14. Note that in any faithful ∇-algebra, $\nabla 1 = 1$. Because, $1 = \nabla \Box 1 = \nabla(1 \to 1) = \nabla 1$.

Corollary 3.15. *Any faithful ∇-algebra is a Heyting algebra.*

Proof. We claim that the Heyting implication is $\nabla(a \to b)$. To prove this claim, if $c \wedge a \leq b$, then by part (i) of Theorem 3.13, we have $\nabla \Box c = c$. Hence, $\nabla \Box c \wedge a \leq b$ which implies $\Box c \leq a \to b$ and then $c = \nabla \Box c \leq \nabla(a \to b)$. Conversely, if $c \leq \nabla(a \to b)$, then $c \wedge a \leq \nabla(a \to b) \wedge a \leq b$. $\Box$

Theorem 3.16. *(Fullness) Let $\mathcal{A} = (A, \nabla, \to)$ be a ∇-algebra. Then, the followings are equivalent:*

(i) $\Box\nabla a = a$, *for any* $a \in A$.

(ii) $\Box$ *is surjective.*

(iii) ∇ *is an embedding, i.e., if* $\nabla a \leq \nabla b$ *then* $a \leq b$.

Proof. See Theorem 2.2. $\Box$

Theorem 3.17. *For any subset* $C \subseteq \{D, N, R, L, Fa, Fu\}$*, the classes* $\mathcal{V}(C)$ *and* $\mathcal{V}_H(C)$ *are varieties. If* $Fa \in C$*, the class* $\mathcal{V}(C, H)$ *is also a variety.*

Proof. For the moment, set $C = \emptyset$ and consider the following set of universal equalities and inequalities:

(1) The set of equalities axiomatizing the variety of bounded lattices,

(2) $a \wedge \nabla(a \to b) \leq b$.

(3) $(a \wedge b) \to a = 1$.

(4) $\nabla(a \wedge b) \leq \nabla a \wedge \nabla b$.

(5) $c \wedge [(\nabla c \wedge a) \to b] \leq a \to b$.

It is easy to see that these axioms can be re-written as equalities. Therefore, it is enough to show that they axiomatize the class of all ∇-algebras. First, we have to show that these axioms are satisfied by any ∇-algebra. The first four are trivially satisfied using the adjunction and the monotonicity of ∇. For the last one, by the adjunction, it is enough to show that $\nabla[c \wedge [(\nabla c \wedge a) \to b]] \wedge a \leq b$. By the monotonicity of ∇, we have

$$\nabla[c \wedge [(\nabla c \wedge a) \to b]] \leq \nabla c \wedge \nabla[(\nabla c \wedge a) \to b]$$

and since $\nabla c \wedge \nabla[(\nabla c \wedge a) \to b] \wedge a \leq b$, we are done.
For the converse, first we need to establish two properties. First, by the axiom (1), we know that A is a bounded lattice. Then, note that $x \leq y$ implies $x \to y = 1$. The reason is that if $x \leq y$, then $x = x \wedge y$ and by the axiom (2), we have $x \to y = (x \wedge y) \to y = 1$. Secondly, by the axioms we know that ∇ is order-preserving, because if $x \leq y$ then $x \wedge y = x$. Hence, $\nabla x = \nabla(x \wedge y) \leq \nabla x \wedge \nabla y$, which implies $\nabla x \leq \nabla y$. Now, we are ready to prove that A is a ∇-algebra. If $\nabla c \wedge a \leq b$ then $(\nabla c \wedge a) \to b = 1$ and hence, by the axiom (5), we have $c \leq a \to b$. Conversely, if $c \leq a \to b$, then since ∇ is order-preserving, $\nabla c \leq \nabla(a \to b)$. Therefore, $\nabla c \wedge a \leq \nabla(a \to b) \wedge a$. By the axiom (4), we reach $\nabla c \wedge a \leq b$.
For the other axioms from the set $\{D, N, R, L, Fa, Fu\}$, using Theorem 3.13

and Theorem 3.16, it is easy to see that each of these axioms can be represented by an equality. Hence, for any $C \subseteq \{D, N, R, L, Fa, Fu\}$, the class $\mathcal{V}(C)$ is a variety. The case for $\mathcal{V}_H(C)$ is also easy, using the fact that being a Heyting algebra is definable by equalities using a new symbol for the Heyting implication. Finally, note that in the presence of (Fa), any ∇-algebra is Heyting and its Heyting implication is definable by $a \supset b = \nabla(a \to b)$, see Corollary 3.15. Hence, when $Fa \in C$, the class $\mathcal{V}(C, H)$ is equal to $\mathcal{V}(C)$ which is proved to be a variety. □

Remark 3.18. Note that Theorem 3.17 excludes the classes $\mathcal{V}(C, H)$, for any $C \subseteq \{D, N, R, L, Fu\}$. The reason is that to make the class $\mathcal{V}(C, H)$ a variety, we need to have access to the Heyting implication in the signature of the algebra to provide some equalities to state that the ∇-algebra is actually Heyting.

4 Subdirectly irreducible and Simple Normal Distributive ∇-algebras

In the realm of universal algebra, subdirectly irreducible and simple algebras are the building blocks to construct all agberas and the simplest possible algebras, respectively, see [24]. In this section, we will focus on these two families in the setting of normal distributive ∇-algebras. To provide a characterization we establish the usual connection between congruence relations and some families of filters.

Let $\mathcal{A}$ be a ∇-algebra. A binary relation $\theta \subseteq A \times A$ is called a *congruence relation* if it is an equivalence, respecting the algebraic operations in the signature, namely the finite meets, the finite joins, ∇ and $\to$. We denote the set of all congruence relations of $\mathcal{A}$ by $\Theta(\mathcal{A})$. Any ∇-algebra has two trivial congruence relations, namely the equality and the whole set $A \times A$. A ∇-algebra is called *simple*, if it has no nontrivial congruence relation and it is called *subdirectly irreducible*, if it is either trivial (with exactly one element) or it has a least non-identity congruence with respect to the inclusion. For more information on the general universal algebraic side see [24] and for the characterization if subdirectly irreducible and simple Heyting algebras see [14].

Remark 4.1. Working with the explicitly Heyting algebras, the previous definition of congruence relations may appear somewhat ambiguous, as it is not clear if it must also respect the Heyting implications. To make the definition more clear, let us emphasize that we do not assume this preservation condition. However, we will see that it automatically follows from the original definition. This makes the definition natural, even in its extended signature.

Definition 4.2. Let $\mathcal{A}$ be a normal ∇-algebra. By a *modal filter* F on $\mathcal{A}$, we mean an upset of $\mathcal{A}$, closed under all finite meets and the modal operators $\Box$ and ∇. We denote the class of all modal filters of $\mathcal{A}$ by $\mathcal{M}(\mathcal{A})$.

Example 4.3. Any normal ∇-algebra has two trivial modal filters $F = \{1\}$ and $F = A$. Note that the normality condition (or at least some part of it) is needed, if we want $F = \{1\}$ to be closed under ∇.

Lemma 4.4. *Let $\mathcal{A}$ be a normal ∇-algebra. For any subset $S \subseteq A$, the least modal filter extending the set S, denoted by $m(S)$, exists and is described by*

$$m(S) = \{y \in A \mid \exists m_i, n_i \in \mathbb{N}\ \exists s_i \in S\ (\bigwedge_i \nabla^{m_i}\Box^{n_i} s_i \leq y)\}.$$

We will denote $m(\{x\})$ by $m(x)$.

Proof. It is clear that any modal filter that extends S includes $m(S)$. Therefore, it is enough to show that $m(S)$ is a modal filter itself. It is clearly a filter and since ∇ commutes with all finite meets, it is also closed under ∇. To show its closure under $\Box$, assume $y \in m(S)$. Hence, there are $m_i, n_i \in \mathbb{N}$ and $s_i \in S$ such that $\bigwedge_i \nabla^{m_i}\Box^{n_i} s_i \leq y$. Therefore, $\Box \bigwedge_i \nabla^{m_i}\Box^{n_i} s_i \leq \Box y$. Since $\Box$ is a right adjoint, it commutes with meets and hence $\bigwedge_i \Box\nabla^{m_i}\Box^{n_i} s_i \leq \Box y$. Define I as the set of all i's such that $m_i > 0$. Then, since for any $x \in A$ we have $x \leq \Box\nabla x$, we have $\nabla^{m_i-1}\Box^{n_i} s_i \leq \Box\nabla^{m_i}\Box^{n_i} s_i$, for any $i \in I$. Hence,

$$(\bigwedge_{i \notin I} \Box^{n_i+1} s_i) \wedge (\bigwedge_{i \in I} \nabla^{m_i-1}\Box^{n_i} s_i) \leq \Box y.$$

Therefore, $\Box y \in m(S)$. $\Box$

In the following, we will show that the modal filters and the congruence relations are in a one-to-one correspondence. To establish that connection, we need the following lemma:

Lemma 4.5. *In any normal distributive ∇-algebra, the following inequalities are satisfied:*

(1) $x \to y \leq (x \wedge z) \to (y \wedge z)$,

(2) $x \to y \leq (x \vee z) \to (y \vee z)$,

(3) $\nabla(x \to y) \leq \nabla x \to \nabla y$,

(4) $\Box(x \to y) \leq (z \to x) \to (z \to y)$,

(5) $\Box(x \to y) \leq (y \to z) \to (x \to z)$.

If the algebra is also Heyting with the Heyting implication $\supset$, we also have:

(6) $x \to y \leq (z \supset x) \to (z \supset y)$,

(7) $x \to y \leq (y \supset z) \to (x \supset z)$.

Proof. For (1), using the adjunction, it is enough to prove $\nabla(x \to y) \wedge x \wedge z \leq y \wedge z$, which is clear from $\nabla(x \to y) \wedge x \leq y$. For (2), using the adjunction, we have to show that $\nabla(x \to y) \wedge (x \vee z) \leq y \vee z$. Using distributivity and the fact that $\nabla(x \to y) \wedge x \leq y$, the claim easily follows. For (3), by normality we have

$$\nabla\nabla(x \to y) \wedge \nabla x = \nabla[\nabla(x \to y) \wedge x] \leq \nabla y.$$

Hence, by adjunction $\nabla(x \to y) \leq \nabla x \to \nabla y$. For (4), by the adjunction, we have to show

$$\nabla\Box(x \to y) \wedge (z \to x) \leq (z \to y).$$

Given the fact that $\nabla\Box(x \to y) \leq (x \to y)$ and that the operation $\to$ is an implication, (see Theorem 3.4), the claim follows. The proof for (5) is similar to that of (4). The cases (6) and (7) are easy. $\Box$

Theorem 4.6. *Let $\mathcal{A}$ be a normal distributive ∇-algebra. Then, there is a one-to-one correspondence between the poset of modal filters of $\mathcal{A}$ and the poset of congruences on $\mathcal{A}$, given by the following operations:*

$$\alpha : \mathcal{M}(\mathcal{A}) \to \Theta(\mathcal{A}), \text{ defined by } \alpha(F) = \{(x, y) \in A^2 \mid x \leftrightarrow y \in F\},$$

$$\beta : \Theta(\mathcal{A}) \to \mathcal{M}(\mathcal{A}), \text{ defined by } \beta(\theta) = \{x \in A \mid (x, 1) \in \theta\}.$$

Proof. First, we show that $\alpha(F)$ is a congruence relation and $\beta(\theta)$ is a modal filter, for any modal filter F and any congruence relation θ. For the former, note that by Theorem 3.4, $\to$ is an implication. Hence, we have $x \to x = 1$ and $(x \to y) \wedge (y \to z) \leq (x \to z)$, for any x, y, and z. Using this fact and the symmetric definition of $\alpha(F)$, it is easy to prove that $\alpha(F)$ is an equivalence relation. To prove that $\alpha(F)$ respects all the operations in the signature, it is enough to use Lemma 4.5 and the fact that F is a filter closed under $\Box$ and ∇. We only prove the hardest case of implication, the rest is similar. To show that $\alpha(F)$ respects the operation $\to$, we prove that $(x, y), (z, w) \in \alpha(F)$ imply $(x \to z, y \to w) \in \alpha(F)$. By definition, as $(x, y), (z, w) \in \alpha(F)$, we have $x \leftrightarrow y, z \leftrightarrow w \in F$. Since F is a filter, $y \to x, z \to w \in F$. Since F is a modal filter, $\Box(y \to x), \Box(z \to w) \in F$. By Lemma 4.5, parts (4) and (5), we have

$$\Box(y \to x) \wedge \Box(z \to w) \leq [(x \to z) \to (y \to z)] \wedge [(y \to z) \to (y \to w)].$$

Since

$$[(x \to z) \to (y \to z)] \wedge [(y \to z) \to (y \to w)] \leq (x \to z) \to (y \to w)$$

and F is a filter, we have $(x \to z) \to (y \to w) \in F$. Similarly, $(y \to w) \to (x \to z) \in F$. Hence, $(x \to z, y \to w) \in \alpha(F)$.
To prove that $\beta(\theta)$ is a modal filter, the only thing to check is the upward-closedness of $\beta(\theta)$. The rest is a clear consequence of the equalities $1 \wedge 1 = \nabla 1 = \Box 1 = 1$. Now, assume $x \leq y$ and $x \in \beta(\theta)$. Since $x \leq y$, we have $x \to y = 1$. Therefore, since θ is a congruence relation and $(x, 1) \in \theta$, we have $(1 \to y, x \to y) \in \theta$. Hence, $(1 \to y, 1) \in \theta$ and then $(\nabla(1 \to y), \nabla 1) \in \theta$. By disjunction with y on both sides, we have $(\nabla(1 \to y) \vee y, \nabla 1 \vee y) \in \theta$. Since $\nabla(1 \to y) \leq y$ and $\nabla 1 = 1$, we have $(y, 1) \in \theta$.
To show that α and β are the inverses of each other, first observe that both α and β clearly preserves the inclusion. To prove $\alpha(\beta(\theta)) = \theta$ and $\beta(\alpha(F)) = F$, we have to show that

$$x \in F \text{ iff } 1 \leftrightarrow x \in F \qquad\qquad (x, y) \in \theta \text{ iff } (x \leftrightarrow y, 1) \in \theta.$$

For the left equivalence, if $x \in F$, since $x \to 1 = 1$, we have $x \to 1 \in F$ and since F is a modal filter $\Box x = 1 \to x \in F$. For the converse, note that if $1 \to x \in F$, then since F is upward-closed and also closed under ∇, using the fact that $\nabla(1 \to x) \leq x$, we have $x \in F$.
For the right equivalence, if $(x, y) \in \theta$, since θ is a congruence relation, we have $(x \leftrightarrow y, x \leftrightarrow x) \in \theta$ which implies $(x \leftrightarrow y, 1) \in \theta$. For the converse, if $(x \leftrightarrow y, 1) \in \theta$, then since $\beta(\theta)$ is a filter, we have $(x \to y, 1) \in \theta$ and hence $(x \wedge \nabla(x \to y), x \wedge \nabla 1) \in \theta$. Therefore, as $x \wedge \nabla(x \to y) = x \wedge y$ and $\nabla 1 = 1$, we have $(x \wedge y, x) \in \theta$. By symmetry, we also have $(x \wedge y, y) \in \theta$ and hence $(x, y) \in \theta$. $\Box$

Remark 4.7. Theorem 4.6 shows that if $\mathcal{A}$ is also a Heyting algebra and θ is a congruence relation, θ must respect the Heyting implication, as well. The reason is that since the congruence relation θ corespondents to a modal filter F, if $(x, y), (z, w) \in \theta$ we have $x \leftrightarrow y, z \leftrightarrow w \in F$. Then, by the parts (6) and (7) in Lemma 4.5, we have

$$(y \to x) \wedge (z \to w) \leq [(x \supset z) \to (y \supset z)] \wedge [(y \supset z) \to (y \supset w)]$$

Since

$$[(x \supset z) \to (y \supset z)] \wedge [(y \supset z) \to (y \supset w)] \leq (x \supset z) \to (y \supset w)$$

we have $(x \supset z) \to (y \supset w) \in F$. Similarly, $(y \supset w) \to (x \supset z) \in F$. Hence, $(x \supset z, y \supset w) \in \theta$. This remark ensures that all the following theorems on congruence extension property, subdirectly irreducible and simple normal distributive ∇-algebras also hold for subdirectly irreducible and simple normal explicitly Heyting ∇-algebras, where the Heyting implication is explicitly mentioned in the signature of the algebras.

Definition 4.8. A class $\mathcal{C}$ of ∇-algebras has the *congruence extension property*, if for any $\mathcal{B}$ in $\mathcal{C}$, any sub-algebra of $\mathcal{B}$ such as $\mathcal{A}$ and any congruence relation θ over $\mathcal{A}$, there exists a congruence relation over $\mathcal{B}$ like ϕ such that $\phi|_A = \theta$.

Corollary 4.9. *The variety $\mathcal{V}(D, N)$ and all of its subclasses have the congruence extension property. The same also holds for $\mathcal{V}_H(N)$.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two normal distributive ∇-algebras, $\mathcal{A}$ be a sub-algebra of $\mathcal{B}$ and θ be a congruence relation over $\mathcal{A}$. Define ϕ on $\mathcal{B}$ by $\{(x, y) \in B^2 \mid x \leftrightarrow y \in m_{\mathcal{B}}(\beta(\theta))\}$, where $\beta(\theta)$ is the corresponding modal filter to θ over $\mathcal{A}$ and $m_{\mathcal{B}}(\beta(\theta))$ is the least modal filter in $\mathcal{B}$ that includes $\beta(\theta)$. By Theorem 4.6, ϕ is a congruence relation over $\mathcal{B}$. Hence, the only thing remained to prove is that $\phi|_A = \theta$. Let $a, b \in A$. Then, we have to show that $(a, b) \in \phi = m_{\mathcal{B}}(\beta(\theta))$ iff $(a, b) \in \theta$. Using Theorem 4.6 for $\mathcal{A}$, the latter is equivalent to $a \leftrightarrow b \in \beta(\theta)$. Now, it is enough to show that $a \leftrightarrow b \in m_{\mathcal{B}}(\beta(\theta))$ iff $a \leftrightarrow b \in \beta(\theta)$. The latter is actually true in a more general form: $c \in m_{\mathcal{B}}(F)$ iff $c \in F$, for any $c \in A$ and any modal filter F over $\mathcal{A}$. One direction is clear. For the other direction, note that if $c \in m_{\mathcal{B}}(F)$, then by Lemma 4.4, there are $m_i, n_i \in \mathbb{N}$ and $a_i \in F$ such that $\bigwedge_i \nabla^{m_i} \Box^{n_i} a_i \leq_{\mathcal{B}} c$. Since, $\mathcal{A}$ is a sub-algebra of $\mathcal{B}$, we have $\bigwedge_i \nabla^{m_i} \Box^{n_i} a_i \leq_{\mathcal{A}} c$ which finally implies $c \in F$. $\square$

Corollary 4.10. *A non-trivial normal distributive ∇-algebra $\mathcal{A}$ is subdirectly irreducible iff there exists $x \in A - \{1\}$ such that for any $y \in A - \{1\}$, there exist $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i} \Box^{n_i} y \leq x$. The same also holds for any $\mathcal{A} \in \mathcal{V}_H(N)$.*

Proof. Using Theorem 4.6, it is enough to prove that the existence of a minimum element between modal filters $F \neq \{1\}$ is equivalent to the existence of $x \in A - \{1\}$ as presented. First, assume that such an x exists. Consider the least modal filter extending x, denoted by $m(x)$. Since $x \neq 1$ and $x \in m(x)$, we know that $m(x) \neq \{1\}$. Therefore, it is enough to show that $m(x) \subseteq F$ for any $F \neq \{1\}$. Since, $F \neq \{1\}$, there is $y \in F$ such that $y \neq 1$. By the condition, there are $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i}\Box^{n_i} y \leq x$. Since, F is a modal filter and $y \in F$, we have $x \in F$. Hence, $m(x) \subseteq F$. Conversely, assume that the minimum element between modal filters $E \neq \{1\}$ exists. Denote this modal filter by F. Then, since $F \neq \{1\}$, there exists $x \in F$ such that $x \neq 1$. Let y be any arbitrary element in $A - \{1\}$. Then, since $y \in m(y)$, we have $m(y) \neq \{1\}$ which by the minimality of F implies $F \subseteq m(y)$. Since $x \in F$ we have $x \in m(y)$. Therefore, by Lemma 4.4, there exist $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i}\Box^{n_i} y \leq x$. $\square$

Corollary 4.11. (*i*) *A non-trivial normal distributive left ∇-algebra $\mathcal{A}$ is subdirectly irreducible iff there exists $x \in A - \{1\}$ such that for any $y \in A - \{1\}$, there exists $k \in \mathbb{N}$ such that $\nabla^k y \leq x$. The same also holds for any $\mathcal{A} \in \mathcal{V}_H(N, L)$.*

(*ii*) *A non-trivial normal distributive right ∇-algebra $\mathcal{A}$ is subdirectly irreducible iff there exists $x \in A - \{1\}$ such that for any $y \in A - \{1\}$, there exists $k \in \mathbb{N}$ such that $\Box^k y \leq x$. The same also holds for any $\mathcal{A} \in \mathcal{V}_H(N, R)$.*

Proof. For (*i*), since in any left ∇-algebra, $z \leq \Box z$ and $\nabla^{k+1} z \leq \nabla^k z$, for any $z \in A$, the condition in Corollary 4.10 is equivalent to the existence of a $k \in \mathbb{N}$ such that $\nabla^k y \leq x$. For (*ii*), use a similar argument, considering the fact that in any right ∇-algebra, $z \leq \nabla z$ and $\Box^{k+1} z \leq \Box^k z$, for any $z \in A$. $\square$

Corollary 4.12. *A non-trivial Heyting algebra $\mathcal{A}$ is subdirectly irreducible iff there exists $x \in A - \{1\}$ such that $y \leq x$, for any $y \in A - \{1\}$.*

Proof. A Heyting algebra is a normal distributive ∇-algebra with $\nabla = \Box = id$. Now, apply Corollary 4.10. $\square$

Corollary 4.13. *A normal distributive ∇-algebra $\mathcal{A}$ is simple iff for any $x \in A - \{1\}$, there exist $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i}\Box^{n_i} x = 0$. The same also holds for any $\mathcal{A} \in \mathcal{V}_H(N)$.*

Proof. Using Theorem 4.6, it is enough to show that the condition in the statement of the corollary is equivalent to the non-existence of the non-trivial modal filters. First, assume that we have the condition and $F \neq \{1\}$ is any arbitrary modal filter. It is enough to show that $F = A$. Since $F \neq \{1\}$, there is $x \in F$ such that $x \neq 1$. By the condition, there are $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i}\Box^{n_i} x = 0$. Since $x \in F$, we have $0 \in F$ and hence $F = A$.

Conversely, assume that $\mathcal{A}$ has no non-trivial modal filters. Let $x \neq 1$. Hence, $m(x) \neq \{1\}$. Therefore, $m(x) = A$, which implies $0 \in m(x)$. By using Lemma 4.4, we finally reach the existence of $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i} \Box^{n_i} x = 0$. $\square$

Corollary 4.14. *(i) A normal distributive left ∇-algebra $\mathcal{A}$ is simple iff for any $x \in A - \{1\}$, there exists $k \in \mathbb{N}$ such that $\nabla^k x = 0$. The same also holds for any $\mathcal{A} \in \mathcal{V}_H(N, L)$.*

(ii) A normal distributive right ∇-algebra $\mathcal{A}$ is simple iff for any $x \in A - \{1\}$, there exists $k \in \mathbb{N}$ such that $\Box^k x = 0$. The same also holds for any $\mathcal{A} \in \mathcal{V}_H(N, R)$.

Proof. For (i), again use the facts $z \leq \Box z$ and $\nabla^{k+1} z \leq \nabla^k z$, for any $z \in A$. For (ii), use $z \leq \nabla z$ and $\Box^{k+1} z \leq \Box^k z$, for any $z \in A$. $\square$

Corollary 4.15. *A Heyting algebra $\mathcal{A}$ is simple iff for any $x \in A$ either $x = 1$ or $x = 0$.*

Theorem 4.16. *There are infinitely many simple finite normal Heyting ∇-algebras.*

Proof. Let $n \geq 1$ be a natural number. Define the topological space $X = \{1, 2, \cdots, n\} \cup \{\omega\}$ with the following topology: A subset $U \subseteq X$ is open iff $U = X$ or $X \subseteq \{1, 2, \cdots, n\}$. Define $f : X \to X$ as the function sending $x \neq n, \omega$ to $x + 1$ and $f(n) = f(\omega) = \omega$. This function is clearly continuous. Now, consider the normal Heyting ∇-algebra $(\mathcal{O}(X), f^{-1})$. We claim that this ∇-algebra is simple. For that purpose, we show that "for any $U \neq X$, we have $\nabla^n U = \emptyset$" and n is the least number with the property presented in the quotation. Let $U \neq X$. Hence, $\omega \notin U$. It is easy to see that for any $m \leq n$, we have $\nabla^m U = \{x \in \{1, 2, \cdots, n\} \mid x + m \in U\}$. Therefore, for $m = n$, we have $\nabla^n U = \emptyset$, while for any $m < n$, we have $\nabla^m \{m + 1\} = \{1\}$. Hence, by Corollary 4.13, the normal distributive ∇-algebra $(\mathcal{O}(X), f^{-1})$ is simple. Finally, note that for different numbers n, this process provides different simple ∇-algebras, because n is uniquely determined from the structure of the ∇-algebra. Hence, we have provided infinitely many finite simple normal Heyting ∇-algebras. $\square$

5 Dedekind-MacNeille Completion

In this section we will show that for any $C \subseteq \{H, N, R, L, Fa, Fu\}$, the variety $\mathcal{V}(C)$ is closed under Dedekind-MacNeille completion. First, let us recall the completion for the posets:

Definition 5.1. Let $(P, \leq)$ be a poset and $S \subseteq P$. By $U(S) = \{x \in P \mid \forall y \in S\ y \leq x\}$ and $L(S) = \{x \in P \mid \forall y \in S\ x \leq y\}$, we mean the set of all upper bounds and all lower bounds of S, respectively. A subset $N \subseteq P$ is called normal iff $LU(N) = N$. Any set in the form $(x]$ is clearly normal. The Dedekind-MacNeille completion of $(P, \leq)$, denoted by $\mathcal{N}(P, \leq)$, is the poset of all normal subsets of P ordered by inclusion. By the canonical map of the completion, we mean $i : (P, \leq) \to \mathcal{N}(P, \leq)$ defined by $i(x) = (x]$.

It is well-known that the poset of all normal subsets is actually a complete lattice with intersection as the meet and $\bigvee_{i \in I} N_i = LU(\bigcup_{i \in I} N_i)$ as the join. Moreover, the canonical map i is an order embedding that preserves all existing meets and joins in $(P, \leq)$ and if the algebra is Heyting, the map also preserves the Heyting implication, see [13]. It is also easy to see that if $\mathcal{A}$ is a bounded lattice itself, any normal subset is actually an ideal. Therefore, as we work exclusively with ∇-algebras which are also bounded lattices, we use the term normal ideals for the normal subsets. The following lemma is a nice characterization of these normal ideals.

Lemma 5.2. *A subset N is normal iff N is representable as an intersection of principal ideals.*

Proof. If N is normal, as $LU(N) = N$ and $LU(N) = \bigcap_{n \in U(N)} (n]$, the claim is clear. Conversely, assume $N = \bigcap_{i \in I} (n_i]$. We show $LU(N) \subseteq N$. The converse, $N \subseteq LU(N)$, always holds. Since $N = \bigcap_{i \in I} (n_i]$, we have $n_i \in U(N)$, because if $x \in N$ then $x \leq n_i$. Now, assume $y \in LU(N)$. Since, $n_i \in U(N)$, we have $y \leq n_i$, for all $i \in I$. This implies $y \in \bigcap_{i \in I} (n_i] = N$. $\square$

Theorem 5.3. *Let $C \subseteq \{H, N, R, L, Fa, Fu\}$. Then, the variety $\mathcal{V}(C)$ is closed under Dedekind-MacNeille completion and the canonical embedding of the completion is also a ∇-algebra embedding. If $H \in C$, the embedding is also a Heyting algebra morphism.*

Proof. First, let us extend the operators ∇ and $\to$, from the ∇-algebra $\mathcal{A} = (A, \nabla, \to)$ to its lattice of normal ideals. For the implication, for any two normal ideals M and N, define $M \to N = \{x \in A \mid \forall m \in M\ \nabla x \wedge m \in N\}$. First, we show that $M \to N$ is normal. Since N is normal, by Lemma 5.2, $N = \bigcap_{i \in I} (n_i]$. Therefore, using the definition, we know that $x \in M \to N$ iff for any $m \in M$ and $i \in I$, we have $\nabla x \wedge m \leq n_i$. Hence, $M \to N = \bigcap_{i \in I, m \in M} (m \to n_i]$ and by Lemma 5.2, $M \to N$ is normal. To show that this implication is an extension of the implication of $\mathcal{A}$, we need to prove:

Claim I. For any $a, b \in A$, $(a] \to (b] = (a \to b]$.

Proof. We have the following series of equivalences:

$$x \in (a] \to (b] \quad \text{iff} \quad \forall y \in (a]\ \nabla x \wedge y \in (b] \quad \text{iff}$$

$$\nabla x \wedge a \leq b \quad \text{iff} \quad x \leq a \to b \quad \text{iff} \quad x \in (a \to b].$$

To extend the ∇ operator, note that the lattice of normal ideals is complete and its meet is the intersection. Moreover, the operation $\Box N = A \to N = \{x \in A \mid \nabla x \in N\}$ on normal ideals clearly preserves the intersections. Hence, by the adjoint functor theorem, Theorem 2.3, $\Box$ has a left adjoint. Call it ∇. This ∇ is also an extension of the ∇ operator on $\mathcal{A}$:

Claim II. For any $a \in A$, $\nabla(a] = (\nabla a]$.

Proof. First, we show that $\nabla(a] \subseteq (\nabla a]$. Note that $a \in \{x \in A \mid \nabla x \leq \nabla a\} = \Box(\nabla a]$. Hence, $(a] \subseteq \Box(\nabla a]$. Since $\nabla \dashv \Box$, we have $\nabla(a] \subseteq (\nabla a]$. Conversely, to prove $(\nabla a] \subseteq \nabla(a]$, since $\nabla(a] \subseteq \nabla(a]$, by $\nabla \dashv \Box$, we have $(a] \subseteq \Box\nabla(a] = \{x \in A \mid \nabla x \in \nabla(a]\}$, which implies $a \in \{x \in A \mid \nabla x \in \nabla(a]\}$, hence $\nabla a \in \nabla(a]$. Therefore, $(\nabla a] \subseteq \nabla(a]$.

Claim III. For any normal ideal N, we have $\nabla N = \bigvee_{n \in N}(\nabla n]$, where $\bigvee$ is the join operator on the lattice of normal ideals, i.e., $\bigvee = LU(\bigcup)$.

Proof. First, we show $\bigvee_{n \in N}(\nabla n] \subseteq \nabla N$. Since $(n] \subseteq N$ and $\nabla(n] = (\nabla n]$, we have $(\nabla n] \subseteq \nabla N$. Hence, $\bigvee_{n \in N}(\nabla n] \subseteq \nabla N$. For the other direction, since $\nabla n \in (\nabla n] \subseteq \bigvee_{n \in N}(\nabla n]$, we have $N \subseteq \{x \in A \mid \nabla x \in \bigvee_{n \in N}(\nabla n]\} = \Box \bigvee_{n \in N}(\nabla n]$, which implies $\nabla N \subseteq \bigvee_{n \in N}(\nabla n]$.

Now we are ready to prove that for any normal ideals M, N, K,

$$\nabla M \cap N \subseteq K \quad \text{iff} \quad M \subseteq N \to K.$$

To prove $(\Rightarrow)$ direction, if $\nabla M \cap N \subseteq K$ and $m \in M$, we have to show $m \in N \to K$ which means that for any arbitrary $n \in N$ we must have $\nabla m \wedge n \in K$. For that purpose, note that $\nabla m \wedge n \in (\nabla m] = \nabla(m] \subseteq \nabla M$ and $\nabla m \wedge n \in N$, and since $\nabla M \cap N \subseteq K$ we have $\nabla m \wedge n \in K$.
Conversely, to prove $(\Leftarrow)$, assume $M \subseteq N \to K$ and $x \in \nabla M \cap N$. Then, we have to show that $x \in K$. Since K is normal, by Lemma 5.2, it is representable as $\bigcap_{i \in I}(k_i]$. Therefore, it is enough to show that $x \leq k_i$, for any $i \in I$. First, since $M \subseteq N \to K$, for any $m \in M$ and $n \in N$, we have $\nabla m \wedge n \in K$ which implies $\nabla m \wedge n \leq k_i$, for any $i \in I$. Consider

$\nabla(x \to k_i)$. Since $x \in N$, for any $m \in M$, we have $\nabla m \wedge x \leq k_i$. Hence, $m \leq x \to k_i$ which implies that $\nabla m \leq \nabla(x \to k_i)$. Therefore, $\nabla(x \to k_i)$ is an upper bound for $\bigcup_{m\in M}(\nabla m]$, hence $\nabla(x \to k_i) \in U(\bigcup_{m\in M}(\nabla m])$. On the other hand, by Claim III, $x \in \nabla M = \bigvee_{m\in M}(\nabla m] = LU(\bigcup_{m\in M}(\nabla m])$, hence $x \leq \nabla(x \to k_i)$. Therefore, $x \leq x \wedge \nabla(x \to k_i) \leq k_i$ which is what we wanted.

Finally, we have to check that the conditions $\{H, N, R, L, Fa, Fu\}$ are preserved by the Dedekind-MacNeille completion. The Heyting case is well-known. For (N), first we prove that $\nabla A = A$. For that purpose, as $\nabla 1 = 1$, we have $A \subseteq (\nabla 1]$. Now, since $\nabla A = \bigvee_{a\in A}(\nabla a]$ we have $A \subseteq \nabla A$. To prove that ∇ commutes with the binary meet, it is enough to show that $\nabla M \cap \nabla N \subseteq \nabla(M \cap N)$, for any two normal ideals M and N. The other direction always holds in a ∇-algebra. Assume $x \in \nabla M \cap \nabla N$. Since $\nabla M = \bigvee_{m\in M}(\nabla m] = LU(\bigcup_{m\in M}(\nabla m])$, "for any y, if $y \geq \nabla m$ for all $m \in M$, we have $x \leq y$". Call the property inside the quotation mark the $(*)$ property for M. The $(*)$ property is also true for N, because we also have $x \in \nabla N$. To prove $x \in \nabla(M \cap N) = LU(\bigcup_{k\in M\cap N}(\nabla k])$, we assume $z \in U(\bigcup_{k\in M\cap N}(\nabla k])$ and we show that $x \leq z$. Since $z \in U(\bigcup_{k\in M\cap N}(\nabla k])$, for any arbitrary $m \in M$ and $n \in N$, we have $z \geq \nabla(m \wedge n)$. Since $\mathcal{A}$ is normal we have $z \geq \nabla m \wedge \nabla n$. Hence, $m \leq \nabla n \to z$ which implies $\nabla m \leq \nabla(\nabla n \to z)$. Therefore, by the $(*)$ property for M, we have $x \leq \nabla(\nabla n \to z)$. Therefore, $\nabla n \wedge x \leq \nabla n \wedge \nabla(\nabla n \to z) \leq z$. Since, $\nabla n \wedge x \leq z$, with similar type of argument as before, we have $\nabla n \leq \nabla(x \to z)$. Hence, by the $(*)$ property for N, we have $x \leq \nabla(x \to z)$ which implies $x \leq z$.
The cases for (L) and (R) are easy. For (Fa), we prove $N \subseteq \nabla\Box N$. If $n \in N$, then by (Fa) for $\mathcal{A}$, we have $n \leq \nabla\Box a$. Hence, $n \in (\nabla\Box n] = \nabla\Box(n] \subseteq \nabla\Box N$. For (Fu), we have to show $\Box\nabla N \subseteq N$. Assume that $x \in \Box\nabla n$. By definition, we have $\nabla x \in \nabla N$. To prove that $x \in LU(N) = N$, it is enough to pick an arbitrary $y \in U(N)$ and show that $x \leq y$. Since $y \in U(N)$, for any $n \in N$, we have $y \geq n$ which implies $\nabla y \geq \nabla n$ and hence $\nabla y \in U(\bigcup_{n\in N}(\nabla n])$. Since $\nabla x \in \nabla N = LU(\bigcup_{n\in N}(\nabla n])$, we have $\nabla x \leq \nabla y$. Since $\mathcal{A}$ satisfies (Fu), the operation ∇ is one-to-one and hence $x \leq y$. Therefore, $x \in LU(N) = N$. □

A Heyting algebra is a right and left distributive ∇-algebra. Hence, By the above theorem, we reprove the following well-known result:

Corollary 5.4. *The variety of all Heyting algebras is closed under Dedekind-MacNeille completion.*

6 Kripke Frames

In this section, we will first recall a variant of intuitionistic Kripke frames, used in [29], [28] and [20]. This variant provides a natural family of ∇-algebras, as explained in [1]. Using the usual prime filter construction, also employed in [1], it is not hard to represent different classes of distributive ∇-algebras by their corresponding classes of intuitionistic Kripke frames. Here, we recall the prime filter construction to expand the characterization of [1] to also cover the case of full and faithful distributive ∇-algebras. The machinery is also required for the duality theory of the next section.

Definition 6.1. Let $(W, \leq)$ be a poset. A tuple $(W, \leq, R)$ is called an intuitionistic Kripke frame or simply a *Kripke frame* if R is compatible with $\leq$, i.e., for any $k, l, k', l' \in W$, if $k' \leq k$, $(k, l) \in R$ and $l \leq l'$, then $(k', l') \in R$. Moreover,

- (N) if there exists an order-preserving function $\pi : W \to W$, called the *normality witness*, such that $(x, y) \in R$ iff $x \leq \pi(y)$, then the Kripke frame is called normal,
- (R) if R is reflexive, the Kripke frame is called right,
- (L) if $R \subseteq \leq$, the Kripke frame is called left,
- (Fa) if for any $x \in W$, there exists $y \in W$ such that $(y, x) \in R$ and for any $z \in W$ such that $(y, z) \in R$ we have $x \leq z$, then the Kripke frame is called faithful,
- (Fu) if for any $x \in W$, there exists $y \in W$ such that $(x, y) \in R$ and for any $z \in W$ such that $(z, y) \in R$ we have $z \leq x$, then the Kripke frame is called full.

For any $C \subseteq \{N, R, L, Fa, Fu\}$, by $\mathbf{K}(C)$, we mean the class of all Kripke frames with the properties described in the set C. For instance, $\mathbf{K}(\{N, Fa\})$ is the class of all normal faithful Kripke frames.
If $\mathcal{K} = (W, \leq, R)$ and $\mathcal{K}' = (W', \leq', R')$ are two Kripke frames, then by a Kripke morphism $f : \mathcal{K} \to \mathcal{K}'$, we mean an order-preserving function from W to W' such that:

- For any $k, l \in W$, if $(k, l) \in R$ then $(f(k), f(l)) \in R'$,
- for any $l' \in W'$ such that $(f(k), l') \in R'$, there exists $l \in W$ such that $(k, l) \in R$ and $f(l) = l'$,

- for any $l' \in W'$ such that $(l', f(k)) \in R'$, there exists $l \in W$ such that $(l, k) \in R$ and $fl \geq' l'$.

If we also have the following condition:

- for any $l' \in W'$ such that $f(k) \leq' l'$, there exists $l \in W$ such that $k \leq l$ and $f(l) = l'$,

then the Kripke morphism f is called a Heyting Kripke morphism. Kripke frames and Kripke morphisms form a category that we loosely denote by its class of objects $\mathbf{K}(C)$. If we use the Heyting Kripke morphisms, instead, then we denote the subcategory by $\mathbf{K}^H(C)$.

Lemma 6.2. *Let $\mathcal{K} = (W, \leq, R)$ be a normal Kripke frame with the normality witness π. Then:*

(i) (R) is equivalent to the condition that $w \leq \pi(w)$, for any $w \in W$,

(ii) (L) is equivalent to the condition that $\pi(w) \leq w$, for any $w \in W$,

(iii) (Fa) is equivalent to the condition that π is an order embedding, i.e., if $\pi(u) \leq \pi(v)$ then $u \leq v$, for any $u, v \in W$,

(iv) (Fu) is equivalent to the surjectivity of π.

Proof. First recall that by normality, the relation $(x, y) \in R$ is equivalent to $x \leq \pi(y)$. We use this equivalence to rewrite all the aforementioned conditions in terms of π. For (i), by normality, it is clear that $(w, w) \in R$ iff $w \leq \pi(w)$ and hence there is nothing to prove. For (ii), note that by normality, the condition (L), i.e., $R \subseteq \leq$ is equivalent to

$$\forall u, v \in W[v \leq \pi(u) \to v \leq u]$$

which is equivalent to $\forall u \in W\ \pi(u) \leq u$. For (iii), assume we have (Fa) and we want to prove that π is an order-embedding. Assume $\pi(u) \leq \pi(v)$ to prove $u \leq v$. By (Fa), there exists $y \in W$ such that $(y, u) \in R$ and for any $z \in W$ if $(y, z) \in R$ then $u \leq z$. By normality, $y \leq \pi(u)$. Set $z = v$. Since $\pi(u) \leq \pi(v)$, we have $y \leq \pi(v)$. Again, by normality, $(y, v) \in R$. Hence, $u \leq v$, by (Fa). For the converse, if π is an order-embedding. To prove (Fa), we have to show that for any $x \in W$, there exists $y \in W$ such that $(y, x) \in R$ and if $(y, z) \in R$ then $x \leq z$. Set $y = \pi(x)$. Since $y \leq \pi(x)$, by normality, $(y, x) \in R$ and if $(y, z) \in R$, meaning $y \leq \pi(z)$, we have $\pi(x) \leq \pi(z)$ which implies $x \leq z$, by the assumption that π is an order-embedding. For (iv), assume (Fu). We show that π is surjective. By (Fu), for any $x \in W$, there exists $y \in W$ such that $(x,y) \in R$ and for any $z \in W$ such that $(z,y) \in R$ we have $z \leq x$. We claim that $\pi(y) = x$. Since $(x,y) \in R$, we have $x \leq \pi(y)$. To show $\pi(y) \leq x$, set $z = \pi(y)$. Then, by normality, $(z,y) \in R$ and hence by (Fu) we have $\pi(y) = z \leq x$. Therefore, $x = \pi(y)$. Conversely, assume π is surjective. We show that (Fu) holds. By surjectivity, for any $x \in W$, there exists $y \in W$ such that $\pi(y) = x$. We claim this y works for the condition (Fu). First, by normality, it is clear that $(x,y) \in R$. Now, if $z \in W$ such that $(z,y) \in R$ then, by normality $z \leq \pi(y) = x$. Hence, $z \leq x$.

□

Lemma 6.3. *Let $\mathcal{K} = (W, \leq, R)$ and $\mathcal{K}' = (W', \leq', R')$ be two normal Kripke frames with the normality witnesses π and π', respectively. Then, for an order-preserving map $f : W \to W'$, the followings are equivalent:*

- (i) *f is a Kripke morphism.*
- (ii) *$f \circ \pi = \pi' \circ f$ and for any $y' \in W'$ such that $(f(x), y') \in R'$, there exists $y \in W$ such that $(x,y) \in R$ and $f(y) = y'$.*
- (iii) *$f \circ \pi = \pi' \circ f$ and $\uparrow \pi'^{-1}(f[U]) = f[\uparrow \pi^{-1}(U)]$, for any upset U of $(W, \leq)$.*

Proof. To prove (ii) from (i), note that the second condition of (ii) is actually the second condition in the definition of Kripke morphisms. Hence, the only thing to prove is $f \circ \pi = \pi' \circ f$. To that purpose, note that by definition, f maps R into R', which in the presence of normality means that if $x \leq \pi(y)$ then $f(x) \leq' \pi'(f(y))$. Hence, by $\pi(k) \leq \pi(k)$, we have $f(\pi(k)) \leq' \pi'(f(k))$. To prove $\pi'(f(k)) \leq' f(\pi(k))$, by the third condition in the definition of a Kripke morphism, we know that for any $l' \in W'$ such that $(l', f(k)) \in R'$, there exists $l \in W$ such that $(l,k) \in R$ and $fl \geq' l'$. Set $l' = \pi'(f(k))$. It is clear that $l' \leq' \pi'(f(k))$ and hence $(l', f(k)) \in R'$. Therefore, there exists $l \in W$ such that $l \leq \pi(k)$ and $l' \leq' f(l)$. Since, f is order-preserving, we have $f(l) \leq' f(\pi(k))$. Therefore, $\pi'(f(k)) = l' \leq' f(\pi(k))$, which implies $\pi'(f(k)) = f(\pi(k))$.
To prove (iii) from (ii), we have to prove the second part of (iii), i.e., $\uparrow \pi'^{-1}(f[U]) = f[\uparrow \pi^{-1}(U)]$, for any upset U in W. Let $l' \in W'$ be an arbitrary element. If we spell out both $l' \in \uparrow \pi'^{-1}(f[U])$ and $l' \in f[\uparrow \pi^{-1}(U)]$, we have to prove:

$$\exists m \leq' l' \, \exists u \in U \; [\pi'(m) = f(u)] \quad \text{iff} \quad \exists l \, \exists v \leq l \; [l' = f(l) \text{ and } \pi(v) \in U]$$

The direction from right to left is clear from the fact that $f \circ \pi = \pi' \circ f$. Because, if there exist l and v such that $v \leq l$, $l' = f(l)$ and $\pi(v) \in U$, then

it is enough to set $u = \pi(v)$ and $m = f(v)$, since $m = f(v) \leq' f(l) = l'$ and $f(u) = f(\pi(v)) = \pi'(f(v)) = \pi'(m)$. For left to right direction, assume $m \leq' l'$, $u \in U$ and $\pi'(m) = f(u)$. Then, since $f(u) = \pi'(m) \leq' \pi'(l')$, we have $(f(u), l') \in R'$. By the second part of (ii), there exists $l \in W$ such that $(u, l) \in R$ and $l' = f(l)$. Therefore, $u \leq \pi(l)$. Since U is an upset, we also have $\pi(l) \in U$. Now it is enough to set $v = l$.
Finally, to prove (i) from (iii), assume $f \circ \pi = \pi' \circ f$ and $\uparrow \pi'^{-1}(f[U]) = f[\uparrow \pi^{-1}(U)]$, for any upset U. To prove that f is a Kripke morphism, we have to check the three conditions in the definition of Kripke morphisms. For the first condition, if $(k, l) \in R$ then $k \leq \pi(l)$ which implies $f(k) \leq' f(\pi(l)) = \pi'(f(l))$. Hence, $(f(k), f(l)) \in R'$. For the second condition, if $(f(k), l') \in R'$, then $f(k) \leq' \pi'(l')$. Set $U =\uparrow k$. Then, since $l' \in \pi'^{-1}(f[U])$ then $l' \in f[\uparrow \pi^{-1}(U)]$. Therefore, there exists $l \in W$and $v \leq l$ such that $l' = f(l)$ and $\pi(v) \geq k$. Hence, $k \leq \pi(l)$ which implies $(k, l) \in R$. For the third condition, if $(l', f(k)) \in R'$, we have $l' \leq \pi'(f(k)) = f(\pi(k))$. Set $l = \pi(k)$. Then, $f(l) \geq' l'$ and $(l, k) \in R$. □

Let $C \subseteq \{N, H, R, L, Fa, Fu\}$. Then, the categories $\mathbf{Alg}_{\nabla}(C, D)$ and $\mathbf{K}(C)^{op}$ are closely related. To present their relationship, we need to provide two functors between them, in reverse directions. For the first one, consider the construction of Example 3.9 that provides an assignment $\mathbf{U}$ defined on objects of $\mathbf{K}^{op}$ by $\mathbf{U}(W, \leq, R) = (U(W, \leq), \nabla_R, \to_R)$, where $\nabla_R(U) = \{x \in X \mid \exists y \in U \; (y, x) \in R\}$ and $U \to_R V = \{x \in X \mid R[x] \cap U \subseteq V\}$ and on morphisms by $\mathbf{U}(f) = f^{-1}$.

Theorem 6.4. *The assignment $\mathbf{U} : \mathbf{K}^{op} \to \mathbf{Alg}_{\nabla}(D)$ is a functor and for any $C \subseteq \{N, H, R, L, Fa, Fu\}$, if $(W, \leq, R) \in \mathbf{K}(C)$, then $\mathbf{U}(W, \leq, R)$ lands in $\mathbf{Alg}_{\nabla}(C, D)$. Moreover, if $H \in C$, the restriction of the functor $\mathbf{U}$ to $[\mathbf{K}^H(C)]^{op}$ lands in $\mathbf{Alg}^H_{\nabla}(C, D)$.*

Proof. First, we study the object part of the functor. Note that $\mathbf{U}(W, \leq, R)$ is a ∇-algebra, as explained in Example 3.9. For distributivity, $U(W, \leq)$ is a locale and hence a Heyting algebra which is also distributive. For $\{N, R, L\}$, see [1]. For (Fa), we have to prove $\nabla \Box U = U$, for any upset $U \subseteq W$. From the adjunction $\nabla \dashv \Box$, it is clear that $\nabla \Box U \subseteq U$. For the converse, assume $x \in U$. Then, since $(W, \leq, R)$ satisfies (Fa), there exists $y \in W$ such that $(y, x) \in R$ and for any $z \in W$ such that $(y, z) \in R$ we have $x \leq z$. Therefore, $y \in \Box U$, because for any $z \in W$ such that $(y, z) \in R$ we have $x \leq z$ and since U is upward-closed and $x \in U$ we have $z \in U$. Now, since $(y, x) \in R$ and $y \in \Box U$, we have $x \in \nabla \Box U$.
For (Fu), we have to prove $\Box \nabla U = U$. Again, from the adjunction $\nabla \dashv \Box$, it is clear that $U \subseteq \Box \nabla U$. For the converse, assume $x \in \Box \nabla U$. Since $(W, \leq, R)$

satisfies (Fu), there exists $y \in W$ such that $(x, y) \in R$ and for any $z \in W$ such that $(z, y) \in R$ we have $z \leq x$. Since $(x, y) \in R$ we have $y \in \nabla U$. Therefore, there exists $w \in W$ such that $(w, y) \in R$ and $w \in U$. Therefore, by (Fu), we have $w \leq x$. Since U is upward-closed, we have $x \in U$.
For the morphisms, we have to prove that if $f : (W, \leq, R) \to (W', \leq', R')$ is a Kripke morphism, then $\mathbf{U}(f) = f^{-1}$ preserves ∇ and the implication and when f is also Heyting, so is $\mathbf{U}(f)$. For ∇, note that

$$x \in f^{-1}(\nabla U) \quad \text{iff} \quad f(x) \in \nabla U \quad \text{iff} \quad \exists y' \in U \ (y', f(x)) \in R'$$

and

$$x \in \nabla f^{-1}(U) \quad \text{iff} \quad \exists y \in f^{-1}(U) \ (y, x) \in R.$$

Therefore, it is enough to show the equivalence between $\exists y' \in U \ (y', f(x)) \in R'$ and $\exists y \in f^{-1}(U) \ (y, x) \in R$. The latter proves the former easily, because $(y, x) \in R$ implies $(f(y), f(x)) \in R'$ and now it is sufficient to set $y' = f(y)$. For the converse, if there exists $y' \in U$ such that $(y', f(x)) \in R'$, then by part (iii) of the definition of Kripke morphisms, there exists $y \in W$ such that $fy \geq' y'$ and $(y, x) \in R$. The only thing to prove is $f(y) \in U$ which is a result of the facts that $y' \in U$, $fy \geq' y'$ and the upward-closedness of U.
The proof for the Heyting implication is well-known and the case for implication is similar to that of Heyting implication.
Finally, we have to show that $\mathbf{U}$ preserves the identity and the composition that is clear by definition. □

Now, it is time to define the second functor that transforms a distributive ∇-algebra to a Kripke frame. For that purpose, we use the usual prime filter construction, extensively explained in [1]. Here, we recall the construction, as we need the detailed explanation to establish the construction for faithful and full distributive ∇-algebras that were missed in [1]. More importantly, the construction plays the main role in the duality theory of the next section and hence deserves a comprehensive presentation.

Prime Filter Construction. Let $\mathcal{A}$ be a distributive ∇-algebra. Define $\mathbf{P}(\mathcal{A}) = (\mathcal{F}_p(\mathcal{A}), \subseteq, R_{\mathcal{A}})$, where $\mathcal{F}_p(\mathcal{A})$ is the set of the prime filters of $\mathcal{A}$ and the relation $R_{\mathcal{A}}$ defined by $(P, Q) \in R_{\mathcal{A}}$ iff $[(a \to b \in P \text{ and } a \in Q) \text{ implies } b \in Q]$, for any $a, b \in A$. Moreover, for any ∇-algebra morphism $f : \mathcal{A} \to \mathcal{B}$ define $\mathbf{P}(f) = f^{-1}$ and set $i_{\mathcal{A}} : \mathcal{A} \to U(\mathcal{F}_p(\mathcal{A}), \subseteq)$ as $i_{\mathcal{A}}(a) = \{P \in \mathcal{F}_p(\mathcal{A}) \mid a \in P\}$.

Lemma 6.5. *$(P, Q) \in R_{\mathcal{A}}$ iff $\nabla[P] = \{\nabla x \mid x \in P\} \subseteq Q$, for any two prime filters P and Q.*

Proof. If $(P, Q) \in R_{\mathcal{A}}$ and $x \in P$, since $x \leq 1 \to \nabla x$ and P is a filter, $1 \to \nabla x \in P$. Therefore, since $1 \in Q$ and $(P, Q) \in R_{\mathcal{A}}$, we reach $\nabla x \in Q$. Conversely, if $\nabla[P] \subseteq Q$, $a \to b \in P$ and $a \in Q$ then $\nabla(a \to b) \in \nabla[P] \subseteq Q$ and since $a \wedge \nabla(a \to b) \leq b$ we have $b \in Q$. □

Theorem 6.6. *The assignment* $\mathbf{P} : \mathbf{Alg}_{\nabla}(D) \to \mathbf{K}^{op}$ *is a functor and* $i_{\mathcal{A}} : \mathcal{A} \to \mathbf{UP}(\mathcal{A})$ *is a* ∇*-algebra embedding, natural in* $\mathcal{A}$*. Moreover, for any* $C \subseteq \{N, H, R, L, Fa, Fu\}$*, if* $\mathcal{A} \in \mathbf{Alg}_{\nabla}(C, D)$*, then* $\mathbf{P}(\mathcal{A})$ *lands in* $\mathbf{K}(C)$*. If* $H \in C$*, the functor* $\mathbf{P}$ *maps* $\mathbf{Alg}^{H}_{\nabla}(C, D)$ *to* $[\mathbf{K}^{H}(C)]^{op}$ *and* $i_{\mathcal{A}}$ *becomes a Heyting morphism.*

Proof. For the sake of readability, we split the proof in some parts.

I. First, note that by Lemma 6.5, it is easy to prove that $R_{\mathcal{A}}$ is compatible with the relation $\subseteq$ which implies that the structure $\mathbf{P}(\mathcal{A})$ is actually a Kripke frame.

II. Here we show that $i_{\mathcal{A}}$ is a ∇-algebra embedding. For that purpose, note that the following three facts are well-known: First, $i_{\mathcal{A}}$ is a bounded lattice embedding, second, it is natural even on all distributive bounded lattices and the third, if $\mathcal{A}$ is a Heyting algebra, $i_{\mathcal{A}}$ also preserves the Heyting implication, see [14], [17]. Therefore, the only thing to check is the preservation of ∇ and $\to$ which are equivalent to the followings:

- $a \to b \in P$ iff for all $Q \in \mathcal{F}_p(\mathcal{A})$ if $(P, Q) \in R_{\mathcal{A}}$ and $a \in Q$ then $b \in Q$,
- $\nabla a \in P$ iff there exists $Q \in \mathcal{F}_p(\mathcal{A})$ such that $(Q, P) \in R_{\mathcal{A}}$ and $a \in Q$.

For the first one, if $a \to b \in P$, $(P, Q) \in R_{\mathcal{A}}$ and $a \in Q$, by the definition of $R_{\mathcal{A}}$, we have $b \in Q$. For the other direction, assume $a \to b \notin P$ and define F as the filter generated by $\nabla[P] \cup \{a\}$ and $I = (b]$. Then, $F \cap I = \emptyset$, because if $x \in F \cap I$, there are $p_1, \cdots, p_n \in P$ such that $\bigwedge_i \nabla p_i \wedge a \leq x \leq b$. Define $p = \bigwedge_i p_i$. Since P is a filter, we have $p \in P$. Since ∇ is increasing, we have $\nabla p \wedge a \leq b$ which implies $p \leq a \to b$. Since $p \in P$ and P is a filter, we have $a \to b \in P$ which is a contradiction. Hence, $F \cap I = \emptyset$. By the prime filter theorem, Theorem 2.4, there exists a prime filter Q such that $F \subseteq Q$ and $Q \cap I = \emptyset$. Therefore, $\nabla[P] \subseteq Q$ which implies $(P, Q) \in R_{\mathcal{A}}$, by Lemma 6.5. Finally, note that we have $a \in Q$ and $b \notin Q$ which contradicts our assumption. Hence, $a \to b \in P$.
For the second one, i.e., the ∇ case, one direction is easy. If there exists $Q \in \mathcal{F}_p(\mathcal{A})$ such that $(Q, P) \in R_{\mathcal{A}}$ and $a \in Q$ we simply have $\nabla a \in P$, because $\nabla[Q] \subseteq P$, by Lemma 6.5. For the converse, if $\nabla a \in P$, define F as the filter generated by $\{a\}$ and set I as the ideal generated by $(\nabla^{-1} P)^c$.

We have $F \cap I = \emptyset$, because if $x \in F \cap I$, there are $y_1, \cdots y_n \in A$ such that $\nabla y_i \notin P$ and $a \leq x \leq \bigvee_i y_i$. Hence, $\nabla a \leq \nabla(\bigvee_i y_i) = \bigvee_i \nabla y_i$. Since $\nabla a \in P$ and P is a filter, we have $\bigvee_i \nabla y_i \in P$. Since P is prime, there is $i \leq n$ such that $\nabla y_i \in P$ which is a contradiction. Hence, $F \cap I = \emptyset$. Therefore, by the prime filter theorem, Theorem 2.4, there exists a prime filter Q such that $F \subseteq Q$ and $Q \cap I = \emptyset$. From the first, we conclude $a \in Q$ and from the second we can prove $Q \subseteq \nabla^{-1} P$ which implies $\nabla[Q] \subseteq P$ and hence $(Q, P) \in R_{\mathcal{A}}$, by Lemma 6.5.

III. Here we check that $\mathbf{P}$ preserves the conditions in the set $\{N, R, L, Fa, Fu\}$. For the conditions $\{N, R, L\}$, we refer the reader to [1]. For the other two conditions, i.e., (Fa) and (Fu), we have the followings:
For (Fa), assume that $\mathcal{A}$ satisfies (Fa). We first show that for any $P \in \mathcal{F}_p(\mathcal{A})$, there exists $Q \in \mathcal{F}_p(\mathcal{A})$ such that $\Box[P] \subseteq Q \subseteq \nabla^{-1}(P)$, where $\Box[P] = \{\Box a \mid a \in P\}$. Define F as the filter generated by $\Box[P]$ and I as the ideal generated by $(\nabla^{-1} P)^c$. We have $F \cap I = \emptyset$. Because, if $x \in F \cap I$, there are $p_1, \cdots, p_n \in P$ and $y_1, \cdots, y_m \in (\nabla^{-1} P)^c$ such that $\bigwedge_i \Box p_i \leq x \leq \bigvee_j y_j$. Define $p = \bigwedge_i p_i$. Since P is a filter, we have $p \in P$. Since $\Box$ commutes with meets, we have $\Box p \leq \bigvee_j y_j$. Hence, $p = \nabla \Box p \leq \bigvee_j \nabla y_j$. Since P is a prime filter, for at least one j we have $\nabla y_j \in P$ which is impossible. Hence, $F \cap I = \emptyset$. Therefore, by the prime filter theorem, Theorem 2.4, there exists a prime filter Q such that $F \subseteq Q$ and $Q \cap I = \emptyset$. By the first, we can prove $\Box[P] \subseteq Q$. By the second we have $Q \subseteq \nabla^{-1}(P)$.
Now, we are ready to prove that $\mathbf{P}(\mathcal{A})$ satisfies the condition (Fa). We have to show that for any prime filter P, there exists a prime filter Q such that $(Q, P) \in R_{\mathcal{A}}$ and for any prime filter M, if $(Q, M) \in R_{\mathcal{A}}$, then $P \subseteq M$. To prove that, pick an arbitrary prime filter P and set Q as the prime filter constructed above. Since $Q \subseteq \nabla^{-1}(P)$, we have $\nabla[Q] \subseteq P$ which implies $(Q, P) \in R_{\mathcal{A}}$, by Lemma 6.5. Let M be a prime filter such that $(Q, M) \in R_{\mathcal{A}}$. Hence, $\nabla[Q] \subseteq M$, by Lemma 6.5. We have to show that $P \subseteq M$. Let $p \in P$. Then $p = \nabla \Box p$. Hence, $p \in \nabla[\Box[P]] \subseteq \nabla[Q] \subseteq M$. Therefore, $P \subseteq M$.
For (Fu), assume that $\mathcal{A}$ satisfies (Fu). We first show that for any $P \in \mathcal{F}_p(\mathcal{A})$, there exists $Q \in \mathcal{F}_p(\mathcal{A})$ such that $\nabla[P] \subseteq Q$ and $\nabla^{-1}(Q) \subseteq P$. Define F as the filter generated by $\nabla[P]$ and I as the ideal generated by $\nabla[P^c]$. We have $F \cap I = \emptyset$. Because, if $x \in F \cap I$, there are $p_1, \cdots, p_n \in P$ and $y_1, \cdots, y_m \in P^c$ such that $\bigwedge_i \nabla p_i \leq x \leq \bigvee_j \nabla y_j$. Define $p = \bigwedge_i p_i$. Since P is a filter, we have $p \in P$. Since ∇ commutes with joins, we have $\nabla p \leq \nabla(\bigvee_j y_j)$. Since $\mathcal{A}$ satisfies (Fu), ∇ is an embedding and hence we have $p \leq \bigvee_j y_j$. Since P is a prime filter, for at least one j, we have $y_j \in P$ which is impossible. Hence, $F \cap I = \emptyset$. Therefore, by the prime filter theorem, Theorem 2.4, there exists a prime filter Q such that $F \subseteq Q$ and $Q \cap I = \emptyset$.

By the first, we can prove $\nabla[P] \subseteq Q$. By the second, we have $\nabla^{-1}(Q) \subseteq P$, because if $x \in \nabla^{-1}(Q)$ then $\nabla x \in Q$. If $x \notin P$, then $\nabla x \in Q \cap \nabla[P^c]$ which is impossible. Hence, $\nabla^{-1}(Q) \subseteq P$.
Now, we can prove that $\mathbf{P}(\mathcal{A})$ satisfies the condition (Fu). We have to show that for any prime filter P, there exists a prime filter Q such that $(P, Q) \in R_{\mathcal{A}}$ and for any prime filter M, if $(M, Q) \in R_{\mathcal{A}}$, then $M \subseteq P$. Let P be an arbitrary prime filter. Pick Q as the prime filter constructed above. Since $\nabla[P] \subseteq Q$, we have $(P, Q) \in R_{\mathcal{A}}$, by Lemma 6.5. Let M be a prime filter such that $(M, Q) \in R_{\mathcal{A}}$. We have to show that $M \subseteq P$. Since $(M, Q) \in R_{\mathcal{A}}$, by Lemma 6.5, we have $\nabla[M] \subseteq Q$ and hence $M \subseteq \nabla^{-1}(Q) \subseteq P$.

IV. Here, we address the morphisms, i.e., we show that if $f : \mathcal{A} \to \mathcal{B}$ is a ∇-algebra morphism, $\mathbf{P}(f) = f^{-1} : (\mathcal{F}_p(\mathcal{B}), \subseteq, R_{\mathcal{B}}) \to (\mathcal{F}_p(\mathcal{A}), \subseteq, R_{\mathcal{A}})$ is a Kripke morphism. To prove that, we have to check the three conditions in the definition of Kripke morphisms.
First, we show that $(P, Q) \in R_{\mathcal{B}}$ implies $(f^{-1}(P), f^{-1}(Q)) \in R_{\mathcal{A}}$, for any $P, Q \in \mathcal{F}_p(\mathcal{B})$. First, note that by the fact that f preserves ∇, we have $\nabla_{\mathcal{A}}[f^{-1}(P)] \subseteq f^{-1}(\nabla_{\mathcal{B}}[P])$. The reason is that if $x \in f^{-1}(P)$, then as $f(\nabla_{\mathcal{A}} x) = \nabla_{\mathcal{B}} f(x) \in \nabla_{\mathcal{B}}[P]$, we have $\nabla_{\mathcal{A}} x \in f^{-1}(\nabla_{\mathcal{B}}[P])$. Now, assume $(P, Q) \in R_{\mathcal{B}}$. By Lemma 6.5, we have $\nabla_{\mathcal{B}}[P] \subseteq Q$ which implies

$$\nabla_{\mathcal{A}}[f^{-1}(P)] \subseteq f^{-1}(\nabla_{\mathcal{B}}[P]) \subseteq f^{-1}(Q).$$

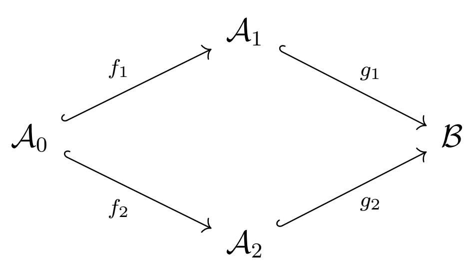
Finally, we reach $(f^{-1}(P), f^{-1}(Q)) \in R_{\mathcal{A}}$, by Lemma 6.5.
For the second condition, assume $(f^{-1}(P'), Q) \in R_{\mathcal{A}}$, for some $P' \in \mathcal{F}_p(\mathcal{B})$ and $Q \in \mathcal{F}_p(\mathcal{A})$. We have to provide $Q' \in \mathcal{F}_p(\mathcal{B})$ such that $(P', Q') \in R_{\mathcal{B}}$ and $f^{-1}(Q') = Q$. Define I as the ideal generated by $f[Q^c]$ and F as the filter generated by $\nabla_{\mathcal{B}}[P'] \cup f[Q]$. We claim that $F \cap I = \emptyset$. Assume $x \in F \cap I$. Then, there exist $y_1, \cdots, y_m \in Q^c$, $z_1, \cdots, z_n \in Q$ and $w_1, \cdots, w_k \in P'$ such that $\bigwedge_j f(z_j) \wedge \bigwedge_r \nabla_{\mathcal{B}} w_r \leq x \leq \bigvee_i f(y_i)$. Define $z = \bigwedge_j z_j$, $w = \bigwedge_r w_r$ and note that $z \in Q$ and $w \in P'$, since both Q and P' are filters. Then, by monotonicity of $\nabla_{\mathcal{B}}$ and the fact that f is a ∇-algebra morphism, we have $\nabla_{\mathcal{B}} w \wedge f(z) \leq f(\bigvee_i y_i)$. Therefore, $w \leq_{\mathcal{B}} f(z) \to_{\mathcal{B}} f(\bigvee_i y_i) = f(z \to_{\mathcal{A}} \bigvee_i y_i)$. Since $w \in P'$, we have $f(z \to_{\mathcal{A}} \bigvee_i y_i) \in P'$ which implies $\nabla_{\mathcal{A}}(z \to_{\mathcal{A}} \bigvee_i y_i) \in \nabla_{\mathcal{A}}[f^{-1}(P')]$. Since $(f^{-1}(P'), Q) \in R_{\mathcal{A}}$, by Lemma 6.5, we have $\nabla_{\mathcal{A}}[f^{-1}(P')] \subseteq Q$. Hence, $\nabla_{\mathcal{A}}(z \to \bigvee_i y_i) \in Q$. Since $z \in Q$ and $z \wedge \nabla_{\mathcal{A}}(z \to_{\mathcal{A}} \bigvee_i y_i) \leq \bigvee_i y_i$, we have $\bigvee_i y_i \in Q$. Since Q is prime, for some i we must have $y_i \in Q$ which is a contradiction with the choice of y_i. Therefore, $F \cap I = \emptyset$. Now, by the prime filter theorem, Theorem 2.4, there exists a prime filter $Q' \in \mathcal{F}_p(\mathcal{B})$ such that $F \subseteq Q'$ and $Q' \cap I = \emptyset$. The first implies $\nabla_{\mathcal{B}}[P'] \cup f[Q] \subseteq Q'$ which also implies $(P', Q') \in R_{\mathcal{B}}$ and $Q \subseteq f^{-1}(Q')$. From the second, we have $f^{-1}(Q') \subseteq Q$, because if $x \in f^{-1}(Q')$

and $x \notin Q$, then $x \in Q^c$ and hence $f(x) \in I$ which implies the contradictory result $f(x) \in Q' \cap I$. Hence, $f^{-1}(Q') = Q$.
For the third condition, let us assume $(Q, f^{-1}(P')) \in R_{\mathcal{A}}$, for some $P' \in \mathcal{F}_p(\mathcal{B})$ and $Q \in \mathcal{F}_p(\mathcal{A})$. We have to find a prime filter $Q' \in \mathcal{F}_p(\mathcal{B})$ such that $(Q', P') \in R_{\mathcal{B}}$ and $Q \subseteq f^{-1}(Q')$. Define I as the ideal generated by $(\nabla_{\mathcal{B}}^{-1}[P'])^c$ and F as the filter generated by $f[Q]$. We have $I \cap F = \emptyset$, because if $x \in I \cap F$, then, there exist $y_1, \cdots, y_m \notin \nabla_{\mathcal{B}}^{-1}[P']$ and $z_1, \cdots, z_n \in Q$ such that $\bigwedge_{i=1}^{n} f(z_i) \leq x \leq \bigvee_{j=1}^{m} y_j$. Define $z = \bigwedge_{i=1}^{n} z_i$ and note that $z \in Q$, since Q is a filter. Since f is a ∇-algebra morphism, $f(z) \leq \bigvee_{j=1}^{m} y_j$. Therefore, $f(\nabla_{\mathcal{A}} z) = \nabla_{\mathcal{B}} f(z) \leq \bigvee_{j=1}^{m} \nabla_{\mathcal{B}} y_j$. Since $(Q, f^{-1}(P')) \in R_{\mathcal{A}}$, by Lemma 6.5, we have $\nabla_{\mathcal{A}}[Q] \subseteq f^{-1}(P')$. Since $z \in Q$, we have $f(\nabla_{\mathcal{A}} z) \in P'$ and hence $\bigvee_{j=1}^{m} \nabla_{\mathcal{B}} y_j \in P'$. Since P' is prime, for some j, we have $\nabla_{\mathcal{B}} y_j \in P'$ which contradicts with $y_j \notin \nabla_{\mathcal{B}}^{-1}[P']$. Hence, $I \cap F = \emptyset$. Therefore, by the prime filter theorem, Theorem 2.4, there exists a prime filter $Q' \in \mathcal{F}_p(\mathcal{B})$ such that $F \subseteq Q'$ and $Q' \cap I = \emptyset$. The first implies that $f[Q] \subseteq Q'$ and hence $Q \subseteq f^{-1}(Q')$ and the second implies $Q' \subseteq \nabla_{\mathcal{B}}^{-1}(P')$ which means $\nabla_{\mathcal{B}}[Q'] \subseteq P'$ and by Lemma 6.5, $(Q', P') \in R_{\mathcal{B}}$.
Lastly, note that if f preserves the Heyting implication, it is a well-known fact that f^{-1} satisfies the fourth condition in the definition of Heyting Kripke morphisms, see [14]. The fact that **P** preserves the identity and composition is clear. □

6.1 Amalgamation Property

In this subsection we use the Kripke representation of distributive ∇-algebras to prove the amalgamation property for the varieties $\mathcal{V}(C, D, N)$ and $\mathcal{V}_H(C, N)$, for any $C \subseteq \{R, L, Fa\}$.

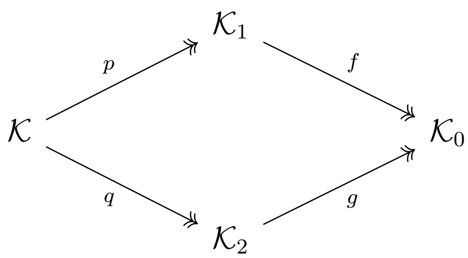
Definition 6.7. A given class $\mathcal{V}$ of ∇-algebras has the *amalgamation property*, if for any $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathcal{V}$ and for any ∇-algebra embeddings $f_1 : \mathcal{A}_0 \longrightarrow \mathcal{A}_1$ and $f_2 : \mathcal{A}_0 \longrightarrow \mathcal{A}_2$, there exist a ∇-algebra $\mathcal{B}$ in $\mathcal{V}$, ∇-algebra embeddings $g_1 : \mathcal{A}_1 \longrightarrow \mathcal{B}$ and $g_2 : \mathcal{A}_2 \longrightarrow \mathcal{B}$ such that $g_1 \circ f_1 = g_2 \circ f_2$:



Lemma 6.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two bounded distributive lattices and $f : \mathcal{A} \to \mathcal{B}$ be an injective lattice map. Then, $f^{-1} : \mathcal{F}_p(\mathcal{B}) \to \mathcal{F}_p(\mathcal{A})$ is surjective. Therefore, the functor $\mathbf{P}$ maps injective ∇-algebra morphisms to surjective Kripke morphisms. Moreover, $\mathbf{U}$ maps surjective Kripke morphisms to injective ∇-algebra morphisms.*

Proof. The last part is an easy consequence of the fact that if $f : W_1 \to W_2$ is surjective, then $\mathbf{U}(f) = f^{-1}$ is one-to-one. For the first part, assume that $f : \mathcal{A} \to \mathcal{B}$ is a one-to-one bounded lattice morphism. Then, we want to prove that for any $P \in \mathcal{F}_p(\mathcal{A})$, there exists a $Q \in \mathcal{F}_p(\mathcal{B})$ such that $f^{-1}(Q) = P$. Define F as the filter generated by $f[P]$ and I as the ideal generated by $f[P^c]$. We have $F \cap I = \emptyset$, because if $x \in F \cap I$, then there exist $p_1, \cdots, p_m \in P$ and $y_1, \cdots y_n \in P^c$ such that $\bigwedge_i f(p_i) \leq x \leq \bigvee_j f(y_j)$. Since f is a bounded lattice morphism, we have $f(\bigwedge_i p_i) \leq f(\bigvee_j y_j)$. Since f is one-to-one, it is an order embedding and hence we have $\bigwedge_i p_i \leq \bigvee_j y_j$. Since P is a prime filter, for at least one j, we must have $y_j \in P$ which contradicts with the choice of y_j. Hence, $F \cap I = \emptyset$. Now, by the prime filter theorem, Theorem 2.4, there exists a prime filter $Q \in \mathcal{F}_p(\mathcal{B})$ such that $F \subseteq Q$ and $Q \cap I = \emptyset$. From the first we have $f[P] \subseteq Q$ which implies $P \subseteq f^{-1}(Q)$ and from the second $f^{-1}(Q) \subseteq P$. Hence, $f^{-1}(Q) = P$. □

Lemma 6.9. *Let $C \subseteq \{R, L, Fa\}$. Then, for any normal Kripke frames $\mathcal{K}_0 = (W_0, \leq_0, R_0)$, $\mathcal{K}_1 = (W_1, \leq_1, R_1)$ and $\mathcal{K}_2 = (W_2, \leq_2, R_2)$ in $\mathbf{K}(C, N)$ and any surjective Kripke morphisms $f : \mathcal{K}_1 \to \mathcal{K}_0$ and $g : \mathcal{K}_2 \to \mathcal{K}_0$, there exists a Kripke frame $\mathcal{K} \in \mathbf{K}(C, N)$ and surjective Kripke morphisms $p : \mathcal{K} \to \mathcal{K}_1$ and $q : \mathcal{K} \to \mathcal{K}_2$ such that $f \circ p = g \circ q$:*



Moreover, if f and g are Heyting, so are p and q.

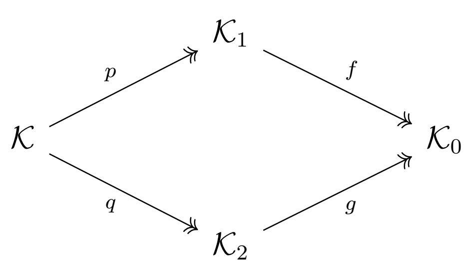
Proof. Define $W = \{(y, z) \in W_1 \times W_2 \mid f(y) = g(z)\}$, $\leq \, = (\leq_1 \times \leq_2)|_W$ and $R = (R_1 \times R_2)|_W$. It is easy to see that $\mathcal{K} = (W, \leq, R)$ is a Kripke frame, as R is clearly compatible with $\leq$. To prove that $\mathcal{K}$ is normal, assume that $\pi_1 : W_1 \to W_1$ and $\pi_2 : W_2 \to W_2$ are the normality witnesses of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Define $\pi : W \to W$ by $\pi(y, z) = (\pi_1(y), \pi_2(z))$. It is well-defined, because $(y, z) \in W$ implies $f(y) = g(z)$ and since f and g are Kripke morphisms, by Lemma 6.3, we have

$$f(\pi_1(y)) = \pi_0(f(y)) = \pi_0(g(z)) = g(\pi_2(z))$$

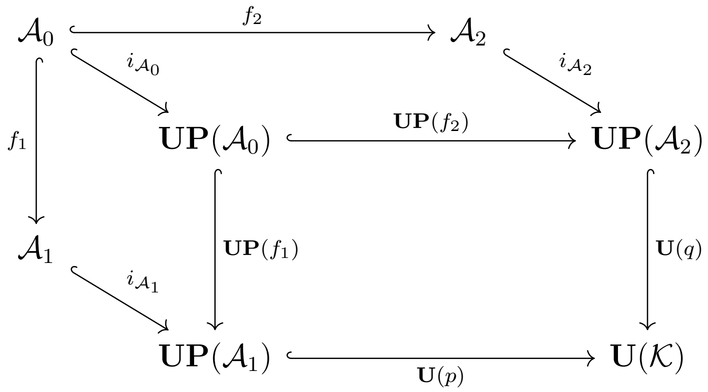
which implies $(\pi_1(y), \pi_2(z)) \in W$. The function π also respects the order $\leq$ and we have $((y, z), (y', z')) \in R$ iff $(y, z) \leq \pi(y, z)$. Hence, $\mathcal{K}$ is normal. To show that $\mathcal{K} \in \mathbf{K}(C)$, using Lemma 6.2, the conditions $\{L, R, Fa\}$ for $\mathcal{K}$ are equivalent to $\pi(y, z) \leq (y, z)$, $(y, z) \leq \pi(y, z)$ and the condition that π is an order embedding, respectively. All the three are inherited from $\mathcal{K}_1$ and $\mathcal{K}_2$ to their product and since they are universal conditions, to the frame $\mathcal{K}$.
Now, we will show that the projections $p : W \to Y$ and $q : W \to Z$ are surjective Kripke morphisms. We prove the claim for p. The case for q is similar. First, note that p is surjective, because for any $y \in W_1$, there exists $z \in W_2$ such that $f(y) = g(z)$ simply because g is surjective. Secondly, p is a Kripke morphism. By the equivalence between (i) and (ii) in Lemma 6.3, it is enough to show that $p \circ \pi = \pi_1 \circ p$ and for any $(y, z) \in W$ and any $y' \in W_1$ such that $(p(y, z), y') \in R_1$, there exists $z' \in Z$ such that $(y', z') \in W$, $((y, z), (y', z')) \in R$. The first condition is clear by the pointwise definition of π. For the second, since $(y, y') = (p(y, z), y') \in R_1$ and f is a Kripke morphism, we have $(f(y), f(y')) \in R_0$. Since $f(y) = g(z)$, we have $(g(z), f(y')) \in R_0$ and as g is a Kripke morphism, there exists $z' \in W_2$ such that $(z, z') \in R_2$ and $f(y') = g(z')$. Therefore, $(y', z') \in W$, $((y, z), (y', z')) \in R$ which completes the proof. The proof to show that p and q are Heyting, if f and g are Heyting is similar. $\square$

Theorem 6.10. *(Amalgamation) Let $C \subseteq \{R, L, Fa\}$. Then, the varieties $\mathcal{V}(C, D, N)$ and $\mathcal{V}_H(C, N)$ have the amalgamation property.*

Proof. Let $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ are ∇-algebras in $\mathcal{V}(C, D, N)$ and $f_1 : \mathcal{A}_0 \longrightarrow \mathcal{A}_1$ and $f_2 : \mathcal{A}_0 \longrightarrow \mathcal{A}_2$ are some embeddings. Applying the functor $\mathbf{P}$, by Theorem 6.6, we reach the normal Kripke models $\mathcal{K}_i = \mathbf{P}(\mathcal{A}_i)$ in $\mathbf{K}(C, N)$, for $i = 0, 1, 2$ and Kripke morphisms $f = \mathbf{P}(f_1) : \mathcal{K}_1 \to \mathcal{K}_0$ and $g = \mathbf{P}(f_2) : \mathcal{K}_2 \to \mathcal{K}_0$. By Lemma 6.8, we know that the maps f and g are surjective. By Lemma 6.9, there exists a Kripke frame $\mathcal{K} \in \mathbf{K}(C, N)$ and surjective Kripke morphisms $p : \mathcal{K} \to \mathcal{K}_1$ and $q : \mathcal{K} \to \mathcal{K}_2$ such that $f \circ p = g \circ q$:



Now, apply the functor $\mathbf{U}$ to the previous diagram and use Thoerem 6.4 to land inside $\mathcal{V}(C, D, N)$. By Lemma 6.8, the maps $\mathbf{U}(p)$ and $\mathbf{U}(q)$ are embeddings. Therefore, as $i_{\mathcal{A}} : \mathcal{A} \to \mathbf{UP}(\mathcal{A})$ is an embedding and also natural in $\mathcal{A}$, we have the following commutative diagram of ∇-algebras in $\mathcal{V}(C, D, N)$:



Now, it is enough to set $\mathcal{A} = \mathbf{U}(\mathcal{K})$ and $g_1 = \mathbf{U}(p) \circ i_{\mathcal{A}_1}$ and $g_2 = \mathbf{U}(q) \circ i_{\mathcal{A}_2}$. Finally, to address the case $\mathcal{V}_H(C, N)$, note that if f_1 and f_2 preserve the Heyting implications, then by Lemma 6.9, the maps p and q will be Heyting Kripke morphisms which implies that g_1 and g_2 preserve Heyting implications. □

7 Logics of Spacetime

In the following, we will introduce some syntactical theories to capture the behavior of different classes of ∇-algebras. Let $\mathcal{L}$ be the usual language of propositional logic plus the modality ∇, i.e., $\mathcal{L} = \{\wedge, \vee, \to, \top, \bot, \nabla\}$. We will define the systems by rules over sequents in the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets and $|\Delta| \leq 1$. Now, consider the following set of rules:

Axioms:

$$\frac{}{A \Rightarrow A} \qquad \frac{}{\Rightarrow \top} \qquad \frac{}{\bot \Rightarrow}$$

Structural Rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\,Lw \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\,Rw \qquad \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\,Lc$$

Cut:

$$\frac{\Gamma \Rightarrow A \qquad \Pi, A \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta}\, cut$$

Conjunction Rules:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\, L\wedge \qquad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\, L\wedge \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\, R\wedge$$

Disjunction Rules:

$$\frac{A \Rightarrow \Delta \qquad B \Rightarrow \Delta}{A \vee B \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\, R\vee \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\, R\vee$$

∇ Rule:

$$\frac{A \Rightarrow B}{\nabla A \Rightarrow \nabla B}\, \nabla$$

Implication Rules:

$$\frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, \nabla(A \to B) \Rightarrow \Delta}\, L \to \qquad \frac{\nabla\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\, R \to$$

Additional Rules:

$$\frac{\Gamma \Rightarrow \Delta}{\nabla\Gamma \Rightarrow \nabla\Delta}\, N \qquad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\, D \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \nabla A \Rightarrow \Delta}\, L$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \nabla A}\, R \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow \nabla(A \to B)}\, Fa \qquad \frac{\nabla\Gamma \Rightarrow A \qquad \nabla\Gamma, B \Rightarrow \nabla\Delta}{\Gamma, A \to B \Rightarrow \Delta}\, Fu$$

Let $C \subseteq \{N, D, R, L, Fa, Fu\}$. By $\mathbf{STL}(C)$, we mean the system consisting of all the above rules, except the additional rules, plus the rules mentioned in C. By $\mathbf{STL}(C, H)$, we mean the rules of $\mathbf{STL}(C)$ plus the usual rules for intuitionistic implication $\supset$, over the extended language $\mathcal{L}_i = \mathcal{L} \cup \{\supset\}$:

Intuitionistic Implication Rules:

$$\frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} L \supset \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} R \supset$$

Remark 7.1. Here are some remarks. For simplicity, we write $A \Leftrightarrow B$ to abbreviate two sequents $A \Rightarrow B$ and $B \Rightarrow A$. First, note that the system $\mathbf{STL}(R, L)$ is nothing but the usual system for intuitionistic logic, **IPC**. The reason is that in the presence of (L) and (R), it is easy to prove that $\nabla A \Leftrightarrow A$ and this fact reduces all the rules of **STL** to the usual rules for intuitionistic logics and specially the rules for $\to$ to the rules of intuitionistic implication. Secondly, note that in the presence of $\supset$, it is easy to prove that $A \to B \Leftrightarrow \top \to (A \supset B)$, using the following proof trees in $\mathbf{STL}(H)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{\nabla(A \to B), A \Rightarrow B} L \to}{\nabla(A \to B), \top, A \Rightarrow B} Lw}{\nabla(A \to B), \top \Rightarrow A \supset B} R \supset}{A \to B \Rightarrow \top \to (A \supset B)} R \to \qquad \dfrac{\dfrac{\Rightarrow \top \qquad \dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \supset B, A \Rightarrow B} L \supset}{\nabla(\top \to (A \supset B)), A \Rightarrow B} L \to}{\top \to (A \supset B) \Rightarrow A \to B} R \to$$

Thirdly, note that in the presence of (Fa), the intuitionistic implication is definable by $\nabla(A \to B)$, because it satisfies both of the rules for intuitionistic implication:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow \nabla(A \to B)} (Fa) \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, \nabla(A \to B) \Rightarrow \Delta} L \to$$

Fourthly, note that in the presence of the rules of **STL**, specially the cut rule, each of the rules in the set $\{N, D, R, L, Fa, Fu\}$ is equivalent to a set of axioms (initial sequents) as will be explained below. These axioms are reminiscent of the algebraic conditions we met before.

As it is well-known, the rule (D) is equivalent to the distributivity axiom $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$. For the rule (N), consider the axioms $\nabla(A \wedge B) \Leftrightarrow \nabla A \wedge \nabla B$ and $\nabla \top \Leftrightarrow \top$. The proof of equivalence is easy and can be found in a more general substructural setting in [1]. However, for the sake of completeness and to show how the rules work, we repeat the proof here. First, note that the axioms are provable by the rule (N). One direction of the axioms are even provable in **STL**:

$$\frac{\Rightarrow \top}{\nabla \top \Rightarrow \top} Lw \qquad \dfrac{\dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A} L\wedge}{\nabla(A \wedge B) \Rightarrow \nabla A} \nabla \qquad \dfrac{\dfrac{B \Rightarrow B}{A \wedge B \Rightarrow B} L\wedge}{\nabla(A \wedge B) \Rightarrow \nabla B} \nabla$$

For the other direction, we have:

$$\dfrac{\Rightarrow \top}{\Rightarrow \nabla\top}\,(N) \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, B \Rightarrow A \wedge B}}{\nabla A, \nabla B \Rightarrow \nabla(A \wedge B)}\,(N)}{\nabla A \wedge \nabla B \Rightarrow \nabla(A \wedge B)}$$

where the double lines mean the existence of an omitted yet easy proof tree. The converse is also true. But first, we need to prove that $\mathbf{STL} \vdash \nabla\bot \Rightarrow \bot$ and $\mathbf{STL} \vdash \nabla(A \vee B) \Rightarrow \nabla A \vee \nabla B$. For the first,

$$\dfrac{\dfrac{\dfrac{\bot \Rightarrow}{\bot \Rightarrow \top \to \bot}\,(Rw)}{\nabla\bot \Rightarrow \nabla(\top \to \bot)}\,\nabla \qquad \dfrac{\Rightarrow \top \qquad \bot \Rightarrow \bot}{\nabla(\top \to \bot) \Rightarrow \bot}\,\to L}{\nabla\bot \Rightarrow \bot}\,cut$$

For the second, for the sake of readability, we prove the apparently general claim that if $\mathbf{STL} \vdash \nabla A \Rightarrow C$ and $\mathbf{STL} \vdash \nabla B \Rightarrow C$, then $\mathbf{STL} \vdash \nabla(A \vee B) \Rightarrow C$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\nabla A \Rightarrow C}{\nabla A, \top \Rightarrow C}\,(Lw)}{A \Rightarrow \top \to C}\,(R \to) \qquad \dfrac{\dfrac{\nabla B \Rightarrow C}{\nabla B, \top \Rightarrow C}\,(Lw)}{B \Rightarrow \top \to C}\,(R \to)}{A \vee B \Rightarrow \top \to C}\,L\vee}{\nabla(A \vee B) \Rightarrow \nabla(\top \to C)}\,\nabla \qquad \dfrac{\Rightarrow \top \qquad C \Rightarrow C}{\nabla(\top \to C) \Rightarrow C}\,L \to}{\nabla(A \vee B) \Rightarrow C}\,cut$$

Then, the claim will be proved, by setting $C = \nabla A \vee \nabla B$. Now, we are ready to prove that the mentioned axioms prove the rule (N). First, note that the axioms easily prove $\bigwedge(\nabla\Gamma) \Rightarrow \nabla(\bigwedge\Gamma)$, for any multiset Γ including $\Gamma = \emptyset$. Therefore,

$$\dfrac{\dfrac{\bigwedge(\nabla\Gamma) \Rightarrow \nabla(\bigwedge\Gamma) \qquad \dfrac{\dfrac{\Gamma \Rightarrow \Delta}{\bigwedge\Gamma \Rightarrow \bigvee\Delta}}{\nabla(\bigwedge\Gamma) \Rightarrow \nabla(\bigvee\Delta)}\,\nabla}{\bigwedge(\nabla\Gamma) \Rightarrow \nabla(\bigvee\Delta)}\,cut}{\nabla\Gamma \Rightarrow \nabla\Delta}$$

The last doubleline uses the fact that ∇ commutes with disjunctions, as we proved above.

The rules (R) and (L) are clearly equivalent to the axioms $\nabla A \Rightarrow A$ and $A \Rightarrow \nabla A$, respectively. For the rule (Fa), we have the axiom $A \Leftrightarrow \nabla(\top \to A)$. First, note that one direction of the axiom, i.e., $\nabla(\top \to A) \Rightarrow A$ is even provable in $\mathbf{STL}$, because

$$\dfrac{\Rightarrow \top \qquad A \Rightarrow A}{\nabla(\top \to A) \Rightarrow A}\,L \to$$

For the other half, consider the following proof tree:

$$\dfrac{\dfrac{A \Rightarrow A}{A, \top \Rightarrow A}\, Lw}{A \Rightarrow \nabla(\top \to A)}\, Fa$$

For the converse, if we have the axiom and $\Gamma, A \Rightarrow B$, then:

$$\dfrac{\dfrac{}{\bigwedge \Gamma \Rightarrow \nabla(\top \to \bigwedge \Gamma)}\, axiom \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \top \quad \bigwedge \Gamma \Rightarrow \bigwedge \Gamma}{\nabla(\top \to \bigwedge \Gamma) \Rightarrow \bigwedge \Gamma}\, L\to \qquad \dfrac{\Gamma, A \Rightarrow B}{\bigwedge \Gamma, A \Rightarrow B}}{\nabla(\top \to \bigwedge \Gamma), A \Rightarrow B}\, cut}{\top \to \bigwedge \Gamma \Rightarrow A \to B}\, R\to}{\nabla(\top \to \bigwedge \Gamma) \Rightarrow \nabla(A \to B)}\, \nabla}{\dfrac{\bigwedge \Gamma \Rightarrow \nabla(A \to B)}{\Gamma \Rightarrow \nabla(A \to B)}}\, cut$$

Finally, the rule (Fu) is equivalent to $A \Leftrightarrow \top \to \nabla A$. First, note that one direction of the axiom, i.e., $A \Rightarrow \top \to \nabla A$ is even provable in **STL**, because:

$$\dfrac{\dfrac{\nabla A \Rightarrow \nabla A}{\nabla A, \top \Rightarrow \nabla A}\, Lw}{A \Rightarrow \top \to \nabla A}\, R\to$$

For the other half, consider the following proof tree:

$$\dfrac{\Rightarrow \top \qquad \nabla A \Rightarrow \nabla A}{\top \to \nabla A \Rightarrow A}\, Fu$$

For the converse, if we have the axiom, we first prove the following claim:

Claim. In the presence of the axioms for (Fu), the sequent $\nabla \Sigma \Rightarrow \nabla \Lambda$ implies $\Sigma \Rightarrow \Lambda$.

Proof of the Claim. Consider the following proof tree and note that $\bigvee \nabla \Lambda \Rightarrow \nabla \bigvee \Lambda$ is easily provable in **STL**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\nabla \Sigma \Rightarrow \nabla \Lambda}{\nabla \Sigma \Rightarrow \bigvee \nabla \Lambda} \qquad \bigvee \nabla \Lambda \Rightarrow \nabla \bigvee \Lambda}{\nabla \Sigma \Rightarrow \nabla \bigvee \Lambda}\, cut}{\{\nabla(\top \to \nabla \sigma)\}_{\sigma \in \Sigma} \Rightarrow \nabla \bigvee \Lambda}\, L\to}{\{\top \to \nabla \sigma\}_{\sigma \in \Sigma} \Rightarrow \top \to \nabla \bigvee \Lambda}\, R\to}{\{\sigma\}_{\sigma \in \Sigma} \Rightarrow \bigvee \Lambda}\, cut \textit{ with the axioms}}{\Sigma \Rightarrow \Lambda}$$

In the proof, the doubleline with the label $L \to$ means applying the rule $L \to$ many times to change any $\nabla\sigma$ to $\nabla(\top \to \nabla\sigma)$ and the doubleline with the label "*cut with the axioms*" means using cut on both sides of the sequent with the axioms $\sigma \Rightarrow \top \to \nabla\sigma$ and $\top \to \nabla \bigvee \Lambda \Rightarrow \bigvee \Lambda$.

Using the claim, it is now easy to prove the rule (Fu) from its corresponding axiom:

$$\dfrac{\dfrac{\nabla\Gamma \Rightarrow A \qquad \nabla\Gamma, B \Rightarrow \nabla\Delta}{\nabla\Gamma, \nabla(A \to B) \Rightarrow \nabla\Delta}\, L \to}{\Gamma, A \to B \Rightarrow \Delta}\, \textit{the claim}$$

Definition 7.2. (Algebraic Semantics) Let $\mathcal{A}$ be a ∇-algebra and $V : \mathcal{L} \to A$ be an assignment, mapping formulas to the elements of $\mathcal{A}$. The pair $(\mathcal{A}, V)$ is called an algebraic model if:

- $V(\top) = 1$ and $V(\bot) = 0$,
- $V(\nabla A) = \nabla V(A)$,
- $V(A \circ B) = V(A) \circ V(B)$, for any $\circ \in \{\wedge, \vee, \to, \supset\}$.

The case for $\supset$ only appears if we work with the language $\mathcal{L}_i$ and $\mathcal{A}$ is Heyting. We say $(\mathcal{A}, V) \vDash \Gamma \Rightarrow \Delta$ when $\bigwedge_{\gamma\in\Gamma} V(\gamma) \leq \bigvee_{\delta\in\Delta} V(\delta)$ and $\mathcal{A} \vDash \Gamma \Rightarrow \Delta$ when $(\mathcal{A}, V) \vDash \Gamma \Rightarrow \Delta$, for all V. If $\mathcal{A} \vDash \Gamma \Rightarrow \Delta$, for any $\mathcal{A}$ in some class $\mathfrak{C}$, we write $\mathfrak{C} \vDash \Gamma \Rightarrow \Delta$.

Theorem 7.3. *(Soundness-Completeness) Let $C \subseteq \{N, H, D, R, L, Fa, Fu\}$. Then,* $\mathbf{STL}(C) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{Alg}_\nabla(C) \vDash \Gamma \Rightarrow \Delta$.

Proof. Both soundness and completeness parts are easy observations and some cases ($\mathbf{STL}(C)$, where $D \in C$ and $C \subseteq \{N, R, L\}$) are extensiviely explained in [1]. We do not spell out the details here, but it is worth giving some remarks. For soundness, first note that all the rules of **STL** are valid in any ∇-algebra. The only non-trivial rules are the implication and the ∇ rule. The first is just the syntactic version of the adjunction $\nabla(-)\wedge a \dashv a \to (-)$, in any ∇-algebra. The second is just the syntactic version of the monotonicity of ∇. For the additional rules, note that by Remark 7.1, it is clear that any rule in the set $\{N, D, R, L, Fa, Fu\}$ is equivalent to a set of axioms that are exactly the ones that define the corresponding ∇-algebras. For instance, the rule (Fa) is equivalent to the axiom $A \Leftrightarrow \nabla(\top \to A)$ over **STL**. Hence, it must be valid in all ∇-algebras in $\mathbf{Alg}_\nabla(Fa)$, because in each of these algebras we have $\nabla\Box a = a$, for any $a \in A$. For completeness, it is enough to construct the Lindenbaum algebra and show that any condition in

$\{N, H, D, R, L, Fa, Fu\}$ is inherited from the logic to its Lindenbaum algebra. Again, the only non-trivial case is the additional rules. As before, by Remark 7.1, we know that the additional rules are equivalent to the axioms over **STL** and as **STL** is valid in all ∇-algebra, the Lindenbaum algebra inherits the property. $\square$

Definition 7.4. (*Topological Frame*) Let X be a topological space and $f : X \to X$ be a continuous map. Recall from Example 3.8 that the tuple (X, f) is called a topological frame. If f is surjective, the topological frame is called faithful and if it is a topological embedding, it is called full. Let $\mathcal{C} \subseteq \{Fa, Fu\}$. Then by $\mathbf{T}(C)$ or $\mathbf{T}(C, H)$, we mean the class of all topological frames with the conditions in C. Note that as $\mathcal{O}(X)$ is a Heyting algebra, the presence of H does not make any ambiguity.

Definition 7.5. (*Topological Semantics*) Let X be a topological space, $f : X \to X$ be a continuous map and $V : \mathcal{L} \to \mathcal{O}(X)$ be an assignment. A tuple (X, f, V) is called a *topological model* if:

- $V(\top) = X$ and $V(\bot) = \emptyset$,
- $V(\nabla A) = f^{-1}(V(A))$,
- $V(A \wedge B) = V(A) \cap V(B)$,
- $V(A \vee B) = V(A) \cup V(B)$,
- $V(A \to B) = f_*[int(V(A)^c \cup V(B))]$,
- $V(A \supset B) = int[V(A)^c \cup V(B)]$.

We say $(X, f, V) \vDash \Gamma \Rightarrow \Delta$ when $\bigcap_{\gamma \in \Gamma} V(\gamma) \subseteq \bigcup_{\delta \in \Delta} V(\delta)$ and $(X, f) \vDash \Gamma \Rightarrow \Delta$ when for all V, $(X, f, V) \vDash \Gamma \Rightarrow \Delta$. If $(X, f) \vDash \Gamma \Rightarrow \Delta$, for any (X, f) in some class $\mathfrak{C}$, we write $\mathfrak{C} \vDash \Gamma \Rightarrow \Delta$.

Let (X, f) be a topological frame. Recall from Example 3.8 that the tuple $(\mathcal{O}(X), f^{-1}, \to_f)$ is a normal ∇-algebra, where $U \to_f V = f_*(U \supset V)$ and $U \supset V = int(U^c \cup V)$ is the Heyting implication of the locale $\mathcal{O}(X)$. Moreover, note that if the space X is T_D, then the topological frame (X, f) is faithful (full) iff the ∇-algebra $(\mathcal{O}(X), f^{-1}, \to_f)$ is faithful (full). Finally, note that a formula A is true in a topological model iff it is true in its corresponding ∇-algebra.

Definition 7.6. (*Kripke Semantics*) Let $(W, \leq, R)$ be a Kripke frame and $V : atoms(\mathcal{L}) \to U(W, \leq)$ be an assignment. Then, the tuple $(W, \leq, R, V)$ is called a *Kripke model*. The forcing relation for atomic formulas, conjunction and disjunction is defined as usual. For ∇, $\to$ and $\supset$, we have:

- $w \Vdash \nabla A$ iff there is $u \in W$ such that $(u, w) \in R$ and $u \Vdash A$,
- $w \Vdash A \to B$ iff for any $u \in W$ such that $(w, u) \in R$, if $u \Vdash A$ then $u \Vdash B$,
- $w \Vdash A \supset B$ iff for any $u \in W$ such that $w \leq u$, if $u \Vdash A$ then $u \Vdash B$.

We say $w \Vdash \Gamma \Rightarrow \Delta$ when $[w \Vdash \bigwedge_{\gamma \in \Gamma} V(\gamma)$ implies $w \Vdash \bigvee_{\delta \in \Delta} V(\delta)]$ and $(W, \leq, R) \vDash \Gamma \Rightarrow \Delta$ when $(W, \leq, R, V), w \Vdash \Gamma \Rightarrow \Delta$, for all $w \in W$ and V. If $(W, \leq, R) \vDash \Gamma \Rightarrow \Delta$, for any $(W, \leq, R)$ in some class $\mathfrak{C}$, we write $\mathfrak{C} \vDash \Gamma \Rightarrow \Delta$.

Lemma 7.7. *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets and $f : P \to Q$ be an order preserving function. Then, if P and Q are equipped with their upset topologies, $f : P \to Q$ is a continuous map and it is a topological embedding iff it is an order embedding.*

Proof. For continuity, note that if U is an upset of $(Q, \leq_Q)$, then $f^{-1}(U)$ is an upset of $(P, \leq_P)$, because if $x \leq_P y$ and $x \in f^{-1}(U)$, then $f(x) \in U$ and since U is an upset, $f(y) \in U$ which implies $y \in f^{-1}(U)$. For the second part, as any upset topology is T_0, by Theorem 2.5, being a topological embedding is equivalent to the surjectivity of f^{-1}. We show that this surjectivity is equivalent to the condition that f is an order embedding. If f^{-1} is surjective, $f(x) \leq_Q f(y)$ and $x \nleq_P y$, then set $U = \uparrow x$ and note that $x \in U$ but $y \notin U$. Since f^{-1} is surjective, there exists $V \subseteq Q$ such that $f^{-1}(V) = U$. Since $x \in U$, we have $f(x) \in V$. Since V is an upset and $f(x) \leq_Q f(y)$, we have $f(y) \in V$ which implies the contradictory consequence that $y \in f^{-1}(V) = U$. Hence, f is an order-embedding. Conversely, if f is an order embedding, first note that $f^{-1}(\uparrow f[U]) = U$. The direction $U \subseteq f^{-1}(\uparrow f[U])$ is trivial. For the converse, if $x \in f^{-1}(\uparrow f[U])$, then by definition, there is $y \in U$ such that $f(y) \leq_Q f(x)$. Since f is an order embedding, $y \leq_P x$ which implies $x \in U$. Therefore, $f^{-1}(\uparrow f[U]) = U$. Finally, since U is arbitrary and $\uparrow f[U]$ is an upset, f^{-1} is proved to be surjective. ☐

Theorem 7.8. *(Strong Completeness)*

(i) (Kripke Frame) Let $C \subseteq \{N, H, R, L, Fa, Fu\}$. Then $\mathbf{STL}(C, D) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{K}(C) \vDash \Gamma \Rightarrow \Delta$.

(ii) (Topological Frames) Let $C \subseteq \{H, Fa, Fu\}$. Then $\mathbf{STL}(C, N, D) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{T}(C) \vDash \Gamma \Rightarrow \Delta$.

(iii) (Complete Lattices) Let $C \subseteq \{N, H, D, R, L, Fa, Fu\}$. Then $\mathbf{STL}(C) \vdash \Gamma \Rightarrow \Delta$ *iff $\Rightarrow \Delta$ is valid in all complete ∇-algebras in* $\mathbf{Alg}_{\nabla}(C)$.

Proof. All the three parts are easy consequences of the embedding theorems we proved in the previous sections. For (i), note that by Theorem 6.6, any distributive ∇-algebra is embedable in a $\mathbf{U}(K, \leq, R))$, for some Kripke frame and this embedding respects the properties in $C \subseteq \{N, H, R, L, Fa, Fu\}$. Since $(K, \leq, R) \vDash \Gamma \Rightarrow \Delta$, iff $\mathbf{U}(K, \leq, R) \vDash \Gamma \Rightarrow \Delta$, the part (i) follows. The part (ii) is proved by part (i) and the observation that the truth of $\Gamma \Rightarrow \Delta$ in a normal Kripke frame $(K, \leq, R)$ is equivalent to its truth in the topological frame (K, π) where K is considered with the upset topology and π is the normality witness of $(K, \leq, R)$. Note that by Lemma 7.7, the function π is continuous, as it is order-preserving and it is a topological embedding iff it is an order embedding. Finally, for (iii), if $D \in C$, use the part (i) and if $D \notin C$, use the Dedekind-McNeille completion. □

Theorem 7.9. *(Deductive Interpolation) Let $C \subseteq \{H, R, L, Fa\}$. Then, the logic* $\mathbf{STL}(N, D, C)$ *enjoys deductive interpolation.*

Proof. The claim is a consequence of Theorem 6.10. □

8 Duality Theory

In this section, we will use the Kripke representation developed in Section 6 to provide the topological dual for some classes of distributive ∇-algebras. The result will generalize both Priestley and Esakia dualities on the one hand and the spectral duality on the other.

8.1 Priestley-Esakia Duality for ∇-algebras

To present a generalization of Priestley and Esakia duality for our generalized notion of implication, let us first briefly recall Priestley and Esakia dualities.

Definition 8.1. A pair $(X, \leq)$ of a topological space and a partial order is called a Priestley space, if X is compact and for any $x, y \in X$, if $x \nleq y$, there exists a clopen upset U such that $x \in U$ and $y \notin U$. A Priestley space is called an Esakia space if the set $\downarrow U = \{x \in X \mid \exists y \in U \; x \leq y\}$ is clopen, for any clopen U. A continuous and order-preserving function $f : (X, \leq_X) \to (Y, \leq_Y)$ between two Priestley spaces is called a Priestley map. In case that both $(X, \leq_X)$ and $(Y, \leq_Y)$ are Esakia spaces and for any $x \in X$ and $y \in Y$ that $f(x) \leq_Y y$, there exists $z \in X$ such that $x \leq_X z$ and $f(z) = y$, the Priestley map f is called an Esakia map. Priestley spaces and Priestley maps constitute a category that we denote by **Pries**. The same also holds for Esakia space and Esakia maps. This category is denoted by **Esakia**.

Lemma 8.2. *([?]) Every Priestley space $(X, \leq)$ has the following properties:*

- *X is a Hausdorff and zero-dimensional space. The latter means that it has a basis consisting of clopen subsets of the space.*
- *For any closed subset $F \subseteq X$, both $\uparrow F$ and $\downarrow F$ are closed. More specifically, $\uparrow x$ is closed, for any $x \in X$.*
- *Any closed upset of X is an intersection of some clopen upsets.*
- *Each open upset of X is a union of clopen upsets of X and each open downset of X is a union of clopen downsets of X.*
- *Each closed upset of X is an intersection of clopen upsets of X and each closed downset of X is an intersection of clopen downsets of X.*
- *Clopen upsets and clopen downsets of X form a subbasis for X.*
- *For each pair of closed subsets F and G of X, if $\uparrow F \cap \downarrow G = \emptyset$, then there exists a clopen upset U such that $F \subseteq U$ and $U \cap G = \emptyset$.*

The Priestley (Esakia) duality defines two functors between the category of bounded distributive algebras (Heyting algebras) and their maps, **DLat** (**Heyting**), and the dual of the category of Priestley spaces and Priestley maps (Esakia spaces and Esakia maps), **Pries** (**Esakia**). First, we have the functor $\mathbf{S} : \mathbf{DLat} \to \mathbf{Pries}^{op}$ defined on objects by $\mathbf{S}(\mathcal{A}) = (\mathcal{F}_p(\mathcal{A}), \subseteq)$, where the topology on $\mathcal{F}_p(\mathcal{A})$ is defined by the basis of the opens in the form $\{P \in \mathcal{F}_p(\mathcal{A}) \mid a \in P \text{ and } b \notin P\}$ and defined on the morphisms by $\mathbf{S}(f) = f^{-1}$. The second functor is $\mathbf{A} : \mathbf{Pries}^{op} \to \mathbf{DLat}$ defined on objects by $\mathbf{A}(X, \leq) = (CU(X, \leq), \subseteq)$, where $CU(X, \leq)$ is the set of all clopen upsets of X and defined on morphisms by $\mathbf{A}(f) = f^{-1}$. These two functors also map the subcategories **Heyting** and $\mathbf{Esakia}^{op}$ to each other.

Theorem 8.3. *(Priestley-Esakia Duality []) The functors $\mathbf{S}$ and $\mathbf{A}$ and the following natural isomorphisms:*

$$\alpha : \mathcal{A} \to \mathbf{A}(\mathbf{S}(\mathcal{A})) \text{ defined by } \alpha(a) = \{P \in \mathcal{F}_p(\mathcal{A}) \mid a \in P\},$$

$$\beta : (X, \leq) \to \mathbf{S}(\mathbf{A}(X, \leq)) \text{ defined by } \beta(x) = \{U \in CU(X, \leq) \mid x \in U\}.$$

establish an equivalence between the categories **DLat** *and* $\mathbf{Pries}^{op}$*. The same also holds for* **Heyting** *and* $\mathbf{Esakia}^{op}$*.*

Lemma 8.4. *Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two Priestley spaces and $f : (X, \leq_X) \to (Y, \leq_Y)$ be a Priestley map. Then:*

(i) *f is surjective iff $f^{-1} : CU(Y, \leq_Y) \to CU(X, \leq_X)$ is one-to-one iff f is an epic map in* **Pries**.

(ii) *f is an order embedding iff $f^{-1} : CU(Y, \leq_Y) \to CU(X, \leq_X)$ is surjective iff f is a regular monic in* **Pries**.

Proof. For (i), first note that if f is surjective, then f^{-1} is clearly one-to-one on all the subsets of Y, including the clopen upsets. For the rest, using Priestley duality, Theorem 8.3, w.l.o.g we can assume that $(X, \leq_X) = (\mathcal{F}_p(\mathcal{A}), \subseteq)$, $(Y, \leq_Y) = (\mathcal{F}_p(\mathcal{B}), \subseteq)$ and $f = \phi^{-1}$, where $\phi : \mathcal{B} \to \mathcal{A}$ is a **DLat** morphism. Then, we have to prove the followings: If ϕ is one-to-one, then ϕ^{-1} is surjective and ϕ is one-to-one iff ϕ is monic. The equivalence part is clear, as in any category of algebraic structures and algebraic morphisms, including **DLat**, monics are the one-to-one homomorphisms, see [**?**]. For the first part, note that if ϕ is one-to-one, then ϕ^{-1} is surjective as proved in Lemma 6.8. For (ii), again by duality, the last equivalence becomes the equivalence between the surjectivity of ϕ and being reqular epic in **DLat** which is true in any category of algebraic structures and algebraic morphisms, including **DLat**, see [**?**]. For the first equivalence, note that if f^{-1} is surjective, then f is clearly an order embedding, because if $f(x) \leq f(y)$ and $x \nleq y$, then there exists a clopen upset $U \subseteq X$ such that $x \in U$ and $y \notin U$. Since f^{-1} is surjective, there exists a clopen upset $V \subseteq Y$ such that $f^{-1}(V) = U$. Therefore, $x \in f^{-1}(V)$ but $y \notin f^{-1}(V)$ which are equivalent to $f(x) \in V$ and $f(y) \notin V$, which is impossible, as V is an upset. For the converse, if f is an order embedding and U is a clopen upset in X, then $f[U]$ is compact and hence closed. Similarly, $f[U^c]$ is also closed. We claim that $\uparrow f[U] \cap \downarrow f[U^c] = \emptyset$. Because, if $y \in \uparrow f[U] \cap \downarrow f[U^c]$, there are $x \in U$ and $z \notin U$ such that $f(x) \leq y \leq f(z)$. Since f is an order embedding, we have $x \leq z$ and since $x \in U$ and U is an upset, we reach $z \in U$ which is a contradiction. Hence, $\uparrow f[U] \cap \downarrow f[U^c] = \emptyset$. Finally, by Lemma 8.2, there exists a clopen upset V such that $f[U] \subseteq V$ and $f[U^c] \cap V = \emptyset$. The latter implies that $f^{-1}(V) \subseteq U$. Hence, $f^{-1}(V) = U$ and f^{-1} is surjective. □

Definition 8.5. A ∇-space is a tuple $(X, \leq, R)$ of a Priestley space $(X, \leq)$ and a binary relation R on X such that:

- R is compatible with the order, i.e., if $x' \leq x$, $(x, y) \in R$ and $y \leq y'$, then $(x', y') \in R$,
- $R[x] = \{y \in X \mid (x, y) \in R\}$ is closed, for every $x \in X$,
- $\Diamond_R(U) = \{x \in X \mid \exists y \in U \ (x, y) \in R\}$ is clopen, for any clopen U,

- $\nabla_R(V) = \{x \in X \mid \exists y \in V \ (y,x) \in R\}$ is a clopen upset, for any clopen upset V.

Note that any ∇-space is a Kripke frame, if we forget the topology of the space. A ∇-space satisfies a condition in the set $\{N, R, L, Fa, Fu\}$, if it satisfies the condition as a Kripke frame. It is called Heyting, if $(X, \leq)$ is an Esakia space.
If $(X, \leq_X, R_X)$ and $(Y, \leq_Y, R_Y)$ are ∇-spaces, by a ∇-space map $f : (X, \leq_X, R_X) \to (Y, \leq_Y, R_Y)$, we mean a Kripke morphism that is also continuous. Note that any ∇-space map is also a Priestley map. A ∇-space map is called Heyting, if it is Heyting as a Kripke morphism. For any $C \subseteq \{N, H, R, L, Fa, Fu\}$, the class of all ∇-spaces satisfying the conditions in C together with ∇-space maps constitute a category that we denote by $\mathbf{Space}_\nabla(C)$. If we restrict the objects to Heyting ∇-spaces and the morphisms to Heyting ∇-space maps, we denote the subcategory by $\mathbf{Space}^H_\nabla(C)$.

Lemma 8.6. *Let $(X, \leq)$ be a Priestley space and $\pi : X \to X$ be a function. Define the relation R on X by $(x, y) \in R$ iff $x \leq \pi(y)$. Then $(X, \leq, R)$ is a normal ∇-space iff π is a Priestley map and $\downarrow \pi[U]$ is clopen, for any clopen U.*

Proof. First note that as $(x, y) \in R$ is equivalent to $x \leq \pi(y)$, we have the following equlities: $R[x] = \pi^{-1}(\uparrow x)$, $\Diamond_R(U) = \downarrow \pi[U]$ and $\nabla_R(V) = \pi^{-1}(V)$, for any $x \in X$, any subset U and any upset V. Now, if $(X, \leq, R)$ is normal, then there exists an order preserving function $\sigma : (X, \leq) \to (X, \leq)$ such that $(x, y) \in R$ iff $x \leq \sigma(y)$. Since this σ is unique, $\sigma = \pi$. Therefore, π is order-preserving and since $\downarrow \pi[U] = \Diamond_R(U)$ is clopen, by definition, the only thing to prove is the continuity of π. To that purpose, we have to show that $\pi^{-1}(U)$ is open, for any open $U \subseteq X$. As the space is Priestley, it has a sub-basis constituting of clopen upsets and clopen downsets, see Lemma 8.2. Since π^{-1} commutes with union, intersection, and complement, it is enough to show that $\pi^{-1}(V)$ is open, for any clopen upset V. But this trivial as $\pi^{-1}(V) = \nabla_R[V]$ and $\nabla_R[V]$ is a clopen upset. Conversely, if π is a Priestley map, then first, it is clear that R is compatible with the order as π is order preserving. Secondly, as $R[x] = \pi^{-1}(\uparrow x)$, by Lemma 8.2 and the fact that π is continuous, the set $R[x]$ is closed. Thirdly, as $\downarrow \pi[U] = \Diamond_R(U)$, for any clopen U, the subset $\Diamond_R(U)$ is clopen. Fourthly, for any clopen upset V, we have $\nabla_R(V) = \pi^{-1}(V)$ which is also a clopen upset, by continuity of π and the fact that π is order-preserving. ∎

Lemma 8.7. *Let $(X, \leq)$ be a Priestley space and $\pi : X \to X$ be a a Priestley map such that $\downarrow \pi[U]$ is clopen, for any clopen U. Define the relation R on X by $(x, y) \in R$ iff $x \leq \pi(y)$. Then:*

(*i*) $(X, \leq, R)$ *is a normal and faithful* ∇*-space iff* π *is a surjective Priestley map iff* π *is an epic map in* **Pries**.

(*ii*) $(X, \leq, R)$ *is a normal and full* ∇*-space iff* π *is a Priestley map that is also an order embedding iff* π *is a regular monic in* **Pries**.

Proof. Using Lemma 6.2, Lemma 8.4 and Lemma 8.6, the proof is clear. □

Lemma 8.6 shows that normal ∇-spaces are uniquely determined with tuples $(X, \leq, \pi)$, where $(X, \leq)$ is a Priestley space and $\pi : (X, \leq) \to (X, \leq)$ is a Priestley map such that $\downarrow \pi[U]$ is clopen, for any clopen U. Here are two remarks. First, note that the last condition is reminiscent of the additional condition on Esakia spaces and this condition actually produces that condition for $\pi = id_X$. Secondly, this characterization of normal ∇-spaces shows that ∇-spaces can be considered as some sort of dynamic Priestley spaces. To complete that topological picture, Lemma 8.7 connects full and faithful ∇-spaces with tuples $(X, \leq, \pi)$, where π is a regular embedding and an epic map in **Pries**, respectively.

Lemma 8.8. *Let* $C \subseteq \{N, H, R, L, Fa, Fu\}$ *and* $\mathcal{X} = (X, \leq, R)$ *be a* ∇*-space. Define* $\nabla_R(U) = \{x \in X \mid \exists y \in U \; (y, x) \in R\}$ *and* $U \to_R V = \{x \in X \mid R[x] \cap U \subseteq V\}$ *over* $CU(X, \leq)$. *Then*

$$\bar{\mathbf{A}}(X, \leq, R) = (CU(X, \leq), \subseteq, \nabla_R, \to_R)$$

is a ∇*-algebra. Moreover, if we define* $\bar{\mathbf{A}}(f) = f^{-1}$*, then* $\bar{\mathbf{A}}$ *defines a functor from* $[\mathbf{Space}_\nabla(C)]^{op}$ *to* $\mathbf{Alg}_\nabla(D, C)$*. The same also holds for* $[\mathbf{Space}^H_\nabla(C)]^{op}$ *and* $\mathbf{Alg}^H_\nabla(D, C)$

Proof. First, note that ∇_R and $\to_R$ are well-defined operations over $CU(X, \leq)$. For ∇_R, since ∇_R maps clopen upsets to themselves, there is nothing to prove. For $\to_R$, note that if U and V are clopen upsets, since $U \cap V^c$ is also clopen and $(U \to_R V)^c = \Diamond_R(U \cap V^c)$, the set $(U \to_R V)^c$ and hence $U \to_R V$ are both clopen. The set $U \to_R V$ is also upward-closed, because if $x \leq y$ and $x \in U \to_R V$, then $R[x] \cap U \subseteq V$. Since R is compatible with the order, $R[y] \subseteq R[x]$, hence $R[y] \cap U \subseteq V$ which implies $y \in U \to_R V$. Now, note that $\bar{\mathbf{A}}(\mathcal{X})$ is a subset of $\mathbf{U}(\mathcal{X})$, restricting from all upsets of $(X, \leq)$ to just clopen upsets, with the same algebraic operations. The reason for ∇_R and $\to_R$ are trivial. For the meet and the joins, it is an easy consequence of the fact that clopen subsets are closed under finite union and intersections. Hence, $\bar{\mathbf{A}}(\mathcal{X})$ inherits the adjunction property $\nabla(-) \wedge a \dashv a \to (-)$ as well as all the universal conditions in $\{N, R, L, Fa, Fu\}$ from $\mathbf{U}(\mathcal{X})$. Moreover, if $f : (X, \leq_X, R_X) \to (Y, \leq_Y, R_Y)$ is a ∇-space map, then it is by definition a

Kripke morphism and hence $\mathbf{U}(f)$ is a ∇-algebra morphism. Therefore, as $\bar{\mathbf{A}}(f) = f^{-1}$ is a restriction of $\mathbf{U}(f)$ from all upsets of $(Y, \leq_Y)$ to just clopen upsets, it must be a ∇-algebra morphism, as well. The only thing to prove is that if $U \subseteq Y$ is a clopen upset, then $f^{-1}(U) \subseteq X$ is also a clopen upset, which is trivial by the continuity and the monotonicity of f. For the Heyting case, if $(X, \leq, R)$ is a Heyting ∇-space, then $(X, \leq)$ is an Esakia space and hence $CU(X, \leq)$ is a Heyting algebra by Esakia duality, Theorem 8.3 and if $f : (X, \leq_X, R_X) \to (Y, \leq_Y, R_Y)$ is a Heyting ∇-space map, we have to show that f^{-1} preserves the Heyting implication, which is trivial by Esakia duality again. □

Lemma 8.9. *Let $C \subseteq \{N, H, R, L, Fa, Fu\}$ and $\mathcal{A} \in \mathbf{Alg}_{\nabla}(D)$. Define $\bar{\mathbf{S}}(\mathcal{A}) = (\mathcal{F}_p(\mathcal{A}), \subseteq, R)$, where $\mathcal{F}_p(\mathcal{A})$ equipped with the Priestley topology as defined by the basis $\{P \in \mathcal{F}_p(\mathcal{A}) \mid a \in P \text{ and } b \notin P\}$ and $(P, Q) \in R$ iff $\nabla[P] \subseteq Q$. Then, $\bar{\mathbf{S}}(\mathcal{A})$ is a ∇-space. Moreover, if we define $\bar{\mathbf{S}}(f) = f^{-1}$, then, $\bar{\mathbf{S}}$ defines a functor from $\mathbf{Alg}_{\nabla}(D, C)$ to $[\mathbf{Space}_{\nabla}(C)]^{op}$. The same also holds for $\mathbf{Alg}^H_{\nabla}(D, C)$ and $[\mathbf{Space}^H_{\nabla}(C)]^{op}$*

Proof. First, since $\mathcal{A}$ is a distributive algebra, by Priestley duality, Theorem 8.3, $(\mathcal{F}_p(\mathcal{A}), \subseteq)$ is a Priestley space. Secondly, note that R is clearly compatible with $\subseteq$. For the rest, recall the lattice isomorphism $\alpha_{\mathcal{A}} : \mathcal{A} \to CU(\mathcal{F}_p(\mathcal{A}), \subseteq)$ defined by $\alpha(x) = \{P \in \mathcal{F}_p(\mathcal{A}) \mid x \in P\}$, see Theorem 8.3. For the second condition of a ∇-space, note that $R[P] = \bigcap_{x \in A} \alpha(\nabla x)$, because $(P, Q) \in R$ iff for any $x \in P$ we have $\nabla x \in Q$. Since each $\alpha(\nabla x)$ is closed in Priestley topology, the set $R[P]$ must be closed, as well. Thirdly, if U is a clopen subset, then there exist some open sets in the basis of Priestley topology in the form $\alpha(a_i) \cap \alpha(b_i)^c$ such that $U = \bigcup_{i \in I} \alpha(a_i) \cap \alpha(b_i)^c$. Since U is closed and a Priestley space is compact and Hausdorff, then U is compact and hence we can assume that I is finite. Then, since $\Diamond_R$ commutes with unions, we have $\Diamond_R(U) = \bigcup_{i \in I} \Diamond_R(\alpha(a_i) \cap \alpha(b_i)^c)$. But, by the canonical construction we observed that α preserves the implication, meaning $\alpha(a_i \to b_i) = \alpha(a_i) \to \alpha(b_i)$. Hence, $[\alpha(a_i \to b_i)]^c = \Diamond_R(\alpha(a_i) \cap \alpha(b_i)^c)$ which is clopen itself. Therefore, since I is finite, $\Diamond_R(U)$ is also clopen. Fourthly, assume V is a clopen upset. Then, by the Priestley duality, Theorem 8.3, there must be $x \in A$ such that $U = \alpha(x)$. As we observed in the canonical construction, α respects ∇, i.e., $\nabla_R(V) = \nabla_R(\alpha(x)) = \alpha(\nabla x)$. Hence, $\nabla_R(V)$ is also a clopen upset.
Finally, for the conditions in the set $\{N, R, L, Fa, Fu\}$, note that the conditions just refer to the Kripke structure and as a Kripke structure, $\mathbf{P}(\mathcal{A})$ and $\bar{\mathbf{S}}(\mathcal{A})$ coincide. For morphisms, We have to prove that $\bar{\mathbf{S}}(f)$ is a continuous Kripke morphism which is a result of Theorem 6.6 and Theorem 8.3. For the Heyting case, if $\mathcal{A}$ is Heyting, then $(\mathcal{F}_p(\mathcal{A}), \subseteq)$ is an Esakia space, by

Theorem 8.3 and if $f : \mathcal{A} \to \mathcal{B}$ preserves the Heyting implication, then f^{-1} is an Esakia map, by Theorem 8.3, again. $\square$

Theorem 8.10. *(Priestley-Esakia duality for distributive ∇-algebras) Let $C \subseteq \{N, H, R, L, Fa, Fu\}$. Then, the functors $\bar{\mathbf{S}}$ and $\bar{\mathbf{A}}$ and the following natural isomorphisms:*

$\alpha : \mathcal{A} \to \bar{\mathbf{A}}(\bar{\mathbf{S}}(\mathcal{A}))$ *defined by* $\alpha(a) = \{P \in \mathcal{F}_p(\mathcal{A}) \mid a \in P\}$,

$\beta : (X, \leq, R) \to \bar{\mathbf{S}}(\bar{\mathbf{A}}(X, \leq))$ *defined by* $\beta(x) = \{U \in CU(X, \leq) \mid x \in U\}$.

establish an equivalence between the categories $\mathbf{Alg}_{\nabla}(D, C) \simeq \mathbf{Space}^{op}_{\nabla}(C)$. *The same also holds for* $\mathbf{Alg}^{H}_{\nabla}(D, C)$ *and* $[\mathbf{Space}^{H}_{\nabla}(C)]^{op}$.

Proof. It is enough to provide two natural isomorphisms to witness $\bar{\mathbf{A}}(\bar{\mathbf{S}}(\mathcal{A})) \simeq \mathcal{A}$ and $\bar{\mathbf{S}}(\bar{\mathbf{A}}(\mathcal{X})) \simeq \mathcal{X}$. For this purpose, use the isomorphisms of the Priestley duality, Theorem 8.3, i.e., $\alpha : \mathcal{A} \to \bar{\mathbf{A}}(\bar{\mathbf{S}}(\mathcal{A}))$ defined by $\alpha(x) = \{P \in \mathcal{F}_p(\mathcal{A}) \mid x \in P\}$ and $\beta : \mathcal{X} \to \bar{\mathbf{S}}(\bar{\mathbf{A}}(\mathcal{X}))$ defined by $\beta(x) = \{U \in CU(\mathcal{X}) \mid x \in U\}$. These are natural isomorphism between the distributive lattice parts and the Priestley space parts and they also works for Esakia spaces and Heyting algebras. The only thing to check is that if α and its inverse also preserve ∇ and $\to$ and if β and its inverse have the three conditions in the definition of a Kripke morphism. For the former, note that α is just $i_{\mathcal{A}}$ in Theorem 6.6, landing in a subalgebra and hence it preserves ∇ and $\to$. Since α is a bijective morphism, its inverse is automatically a ∇-algebra morphism and hence there is nothing to prove. For the latter, let $\mathcal{X} = (X, \leq, R)$. We show that $(x, y) \in R$ iff $(\beta(x), \beta(y)) \in R_{\bar{\mathbf{A}}(\mathcal{X})}$, which clearly implies all the three conditions, both for β and its inverse. Spelling out the details, $(\beta(x), \beta(y)) \in R_{\bar{\mathbf{A}}(\mathcal{X})}$ is equivalent to $\nabla_R[\beta(x)] \subseteq \beta(y)$ which is equivalent to the following statement: For any clopen upset U of $\mathcal{X}$, if $x \in U$ then $y \in \nabla_R(U)$. Now, we have to prove that this condition is equivalent to $(x, y) \in R$. For the first direction, if $(x, y) \in R$, then if $x \in U$, by the definition of ∇_R, we have $y \in \nabla_R(U)$. For the converse, consider the closed upset $R[x]$. Since $(X, \leq)$ is a Priestley space, by Lemma 8.2, the set $R[x]$ is an intersection of a family of clopen upsets V_i, i.e., $R[x] = \bigcap_{i \in I} V_i$. For any V_i, define W_i as $\{z \in X \mid R[z] \subseteq V_i\}$. Since V_i is a clopen upset and $W_i = \Box_R V_i$, the set W_i is also a clopen upset of X. Since $R[x] \subseteq V_i$, we have $x \in W_i$. Therefore, by the condition we have $y \in \nabla_R(W_i)$ which means that there exists $z \in W_i$ such that $(z, y) \in R$. Since $z \in W_i$, by the definition of W_i we have $R[z] \subseteq V_i$. Therefore, $y \in V_i$. Finally, since $R[x] = \bigcap_{i \in I} V_i$, we have $y \in R[x]$ which implies $(x, y) \in R$. $\square$

8.2 Spectral Duality for ∇-algebras

In this subsection we rephrase the Priestley-Esakia duality of the previous subsection as a spectral duality theorem. As before, first let us recall the well-known isomorphism between the category of Priestly spaces and the category of spectral spaces and what it induces to capture the Esakia spaces.

Definition 8.11. A topological space is called spectral if it is sober and compact and the set of its compact open subsets forms a topological base. A continuous map $f : X \to Y$ between spectral spaces is called spectral, if $f^{-1}(U)$ is compact, for any compact open U. A subset $Y \subseteq X$ is called spectral, if it is a spectral space with the subspace topology and the inclusion map $i : Y \to X$ is a spectral map. It is called doubly spectral, if both Y and Y^c are spectral subsets of X. See [7]. We denote the resulting category of spectral spaces and spectral maps by **Spec**.

It is a well-known fact that the category **Pries** is isomorphic to the category **Spec**, via the following functors:

Theorem 8.12. *([]) Consider the following functors:*

> F : **Pries** $\to$ **Spec** *defined by* $F(X, \leq) = X^s$ *on objects, where* X^s *is the set* X *with the spectral topology, i.e., the topology consisting of all the open upsets of* $(X, \leq)$*. The functor* F *is defined as identity on the morphisms,*
>
> G : **Spec** $\to$ **Pries** *defined by* $G(X) = (X^p, \leq)$ *on objects, where* X^p *is the set* X *with the patch topology, i.e., the topology generated by the basis elements of the form* $U \cap V^c$*, where* U *and* V *are open subsets of* X *and* $\leq$ *is the specialization order of the original topology of* X*, i.e.,* $x \leq y$ *iff* $x \in Cl(\{y\})$*, where* Cl *is the closure operator for the topology of* X*. The functor* G *is defined as identity on the morphisms.*

Then, F *and* G *establish an isomorphism between the categories* **Pries** *and* **Spec**.

Definition 8.13. Let X be a topological space. A subset $Y \subseteq X$ is called saturated, if it is an intersection of some open subsets and it is called co-saturated if it is a union of some closed subsets.

Theorem 8.14. *([7]) Let* $(X, \leq)$ *be a Priestley space and* X^s *be its corresponding spectral space. Then, we have the following dictionary:*

- *a subset of* X *is an upset in the Priestley space* $(X, \leq)$ *iff it is saturated in* X^s*,*

- *a subset of X is a downset in the Priestley space $(X, \leq)$ iff it is co-saturated in X^s,*

- *a subset of X is open in the Priestley space $(X, \leq)$ iff its complement is a spectral subset of X^s,*

- *a subset of X is closed in the Priestley space $(X, \leq)$ iff it is a spectral subset of X^s,*

- *a subset of X is clopen upset of the Priestley space $(X, \leq)$ iff it is compact open in X^s,*

- *a subset of X is closed upset in the Priestley space $(X, \leq)$ iff it is compact and saturated in X^s,*

- *for any $Y \subseteq X$, we have $Cl(Y) = \downarrow Cl_p(Y)$, where Cl and Cl_p are the closure operators for the spectral topology and the Priestley topology, respectively.*

Definition 8.15. ([7]) Let X be a spectral space. It is called Heyting, if $Cl(Y)$ is a doubly spectral subset of X, for any doubly spectral subset Y. A spectral map $f : X \to Y$ between two Heyting spectral spaces is called Heyting, if for any $x \in X$ we have $f(Sat_X(x)) = Sat_Y(f(x))$, where for any spectral space Z and any element $z \in Z$, by $Sat_Z(z)$ we mean $\bigcap_{z \in U} U$.

Theorem 8.16. *([7]) The categories* $\mathbf{Spec}^{op}$ *and* **DLat** *are equivalent. The same also holds for* $[\mathbf{Spec}^H]^{op}$ *and* **Heyting**

Lemma 8.17. *Let X and Y be two spectral spaces and $f : X \to Y$ be a spectral map. Then:*

(i) f is surjective iff f is an epic map in **Spec**.

(ii) f is a topological embedding iff f is a regular monic in **Spec**.

Proof. (i) is an easy consequence of the isomorphism between **Pries** and **Spec**, together with Lemma 8.4. For (ii), as spectral spaces are T_0, by Theorem 2.5, it is enough to prove the equivalence between the surjectivity of $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ and being regular monic in **Spec**. If $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective, we prove that f is an order embedding with respect to the specialization order. If $f(x) \leq_Y f(y)$, then by definition, for any open $U \in Y$, if $f(x) \in U$ then $f(y) \in U$, or equivalently, $x \in f^{-1}(U)$ implies $y \in f^{-1}(U)$. Since f^{-1} is surjective, we have $x \in V$ implies $y \in V$, for any open $V \subseteq X$. Hence, $x \leq_X y$, by definition. Therefore, by Lemma 8.4, f is regular monic in **Pries** and hence in **Spec**. Conversely, if f is regular monic in **Spec**, then it is regular monic in **Pries** by Theorem 8.12. If $(X, \leq_X)$ and $(Y, \leq_Y)$ are the corresponding Priestly spaces for the spectral spaces X and Y, respectively, then by Lemma 8.4, $f^{-1} : CU(Y, \leq_Y) \to CU(X, \leq_X)$ is surjective. Now, by Theorem 8.12, the open sets in the spectral topology are the open upsets of $(X, \leq_X)$ and $(Y, \leq_Y)$. Therefore, it is enough to show that the surjectivity of $f^{-1} : CU(Y, \leq_Y) \to CU(X, \leq_X)$ implies the surjectivity of $f^{-1} : OU(Y, \leq_Y) \to OU(X, \leq_X)$, where $OU(-)$ means the set of all open upsets. This is clear as any open upset of $(X, \leq_X)$ is a union of clopen upsets by Lemma 8.2. □

Definition 8.18. A ∇-spectral space is a pair (X, R) of a spectral space X and a binary relation R on X such that:

- $R[x] = \{y \in X \mid (x, y) \in R\}$ is compact and saturated, for every $x \in X$,
- $R^{-1}[y] = \{x \in X \mid (x, y) \in R\}$ is co-saturated, for every $y \in X$,
- $\Diamond_R(Y) = \{x \in X \mid \exists y \in Y \ (x, y) \in R\}$ is a doubly spectral subset of X, for any doubly spectral subset Y,
- $\nabla_R(V) = \{x \in X \mid \exists y \in V \ (y, x) \in R\}$ is a compact open subset, for any compact open subset V.

Let (X, R) be a spectral ∇-algebra. Then:

(N) if there exists a continuous map $\pi : X \to X$, called the normality witness, such that $(x, y) \in R$ iff $x \in Cl(\pi(y))$, then the ∇-spectral space is called normal,

(H) if $Cl(Y)$ is a doubly spectral subset of X, for any doubly spectral subset Y is called Heyting,

(R) if R is reflexive, the ∇-space is called right,

(L) if $(x, y) \in R$ implies $x \in Cl(\{y\})$, for any $x, y \in X$, the ∇-space is called left,

(Fa) if for any $x \in W$ there exists $y \in W$ such that $(y, x) \in R$ and for any $z \in W$ such that $(y, z) \in R$ we have $x \in Cl(\{z\})$, then the ∇-spectral space is called faithful.

(Fu) if for any $x \in W$ there exists $y \in W$ such that $(x, y) \in R$ and for any $z \in W$ such that $(z, y) \in R$ we have $z \in Cl(\{x\})$, then the ∇-spectral space is called full.

If (X, R_X) and (Y, R_Y) are two ∇-spectral spaces, by a ∇-spectral map $f : (X, R_X) \to (Y, R_Y)$, we mean a spectral map $f : X \to Y$ such that:

- for any $x, x' \in X$, if $(x, x') \in R_X$ then $(f(x), f(x')) \in R_Y$,
- for any $y \in Y$ such that $(f(x), y) \in R_Y$, there exists $z \in X$ such that $(x, z) \in R_X$ and $fz = y$,
- for any $y \in Y$ such that $(y, f(x)) \in R_Y$, there exists $z \in X$ such that $(z, x) \in R_X$ and $y \in Cl_Y(\{f(z)\})$.

A ∇-spectral map $f : (X, R_X) \to (Y, R_Y)$ is called Heyting, if it is Heyting as a spectral map. For any $C \subseteq \{N, H, R, L, Fa, Fu\}$, the class of all ∇-spectral spaces satisfying the conditions in C together with ∇-spectral maps constitute a category that we denote by $\mathbf{Spec}_{\nabla}(C)$. If we restrict the objects to Heyting ∇-spectral spaces and the morphisms to Heyting ∇-spectral maps, we denote the subcategory by $\mathbf{Spec}^H_{\nabla}(C)$.

Lemma 8.19. *Let $C \subseteq \{N, H, R, L, Fa, Fu\}$, X be a spectral space and $(X^p, \leq)$ be its corresponding Priestley space. Then, $(X, R) \in \mathbf{Spec}_{\nabla}(C)$ iff $(X^p, \leq_X, R) \in \mathbf{Space}_{\nabla}(C)$, for any binary relation R on X. The same also holds for $\mathbf{Spec}^H_{\nabla}(C)$ and $\mathbf{Space}^H_{\nabla}(C)$. Therefore, the isomorphism in Theorem 8.12 extends to an isomorphism between $\mathbf{Spec}_{\nabla}(C)$ and $\mathbf{Space}_{\nabla}(C)$. Moreover, we have $\mathbf{Spec}^H_{\nabla}(C) \simeq \mathbf{Space}^H_{\nabla}(C)$.*

Proof. Using the dictionary in Lemma 8.14, it is clear that the conditions in the definition of a spectral ∇-space is equivalent to the definition of a ∇-space. For instance, the conditions that $R[x]$ is compact and saturated and $R^{-1}[y]$ is co-saturated in X is equivalent to the condition that $R[x]$ is an closed upset and $R^{-1}[y]$ is a downset that is equivalent to the conditions that R is compatible with the order and $R[x]$ is closed. For the conditions in $\{R, L, Fa, Fu\}$, there is nothing to prove. For (N), if $(X^p, \leq_X, R)$ is a normal ∇-space, then its normality witness π is a Priestley map by Lemma 8.6 and hence by Theorem 8.12, it is spectral and more specifically continuous in spectral topology. Conversely, if (X, R) is a ∇-spectral space, then its normality witness π is continuous with respect to spectral topology and hence it preserves the specialization order which implies that $(X^p, \leq_X, R)$ is a normal ∇-space. For (H), see [7]. For morphisms, there is nothing to prove, according to Theorem 8.12. For Heyting morphisms, see [7]. $\square$

Lemma 8.20. *Let X be a spectral space and $\pi : X \to X$ be a function. Define the relation R on X by $(x, y) \in R$ iff $x \in Cl(\pi(y))$. Then (X, R) is a normal ∇-spectral space iff π is a spectral map and $Cl(\pi[Y])$ is a doubly spectral subset, for any doubly spectral subset Y.*

Proof. Let $(X^p, \leq)$ be the corresponding Priestley space. First, note that by Lemma 8.19, (X, R) is a normal ∇-spectral space iff $(X^p, \leq, R)$ is a normal Priestley space. We will show that "π is a spectral map and $Cl(\pi[Y])$ is a doubly spectral subset, for any doubly spectral subset Y" is equivalent to "π is a Priestley map and $\downarrow \pi[V]$ is clopen, for any clopen V". Then, the claim is a consequence of Lemma 8.6. For the equivalence, using the dictionary in Lemma 8.14, the only thing to prove is the equivalence of the second parts, for any Priestley map π. To that purpose, we first show that $\downarrow \pi[U] = Cl(\pi[U])$, for any clopen U in the Priestley topology, where Cl is the closure operator in the spectral topology. Since U is closed in Priestley topology, it is compact with that topology and since π is Priestley, $\pi[U]$ is also compact with Priestley topology and hence closed in that topology. By the last part of Lemma 8.14, we have $Cl(\pi[U]) = \downarrow Cl_p(\pi[U])$, where Cl and Cl_p are the closure operators for the spectral topology and the Priestley topology, respectively. Since $\pi[V]$ is closed in Priestley topology, we have $Cl_p(\pi[U]) = \pi[U]$. Hence, $Cl(\pi[U]) = \downarrow \pi[U]$. Now, using the dictionary in Lemma 8.14, we know that a subset U is clopen in Priestley topology iff it is doubly spectral in spectral topology. This completes the proof. □

Lemma 8.21. *Let X be a spectral space and $\pi : X \to X$ be a spectral map such that $Cl(\pi[Y])$ is doubly spectral subset, for any doubly spectral subset Y. Define the relation R on X by $(x, y) \in R$ iff $x \in Cl(\pi(y))$. Then:*

(i) *(X, R) is a normal and faithful ∇-spectral space iff π is a surjective spectral map iff π is an epic map in* **Spec**.

(ii) *(X, R) is a normal and full ∇-spectral space iff π is a spectral map that is also a topological embedding iff π is a regular monic in* **Spec**.

Proof. Using Lemma 8.20, Lemma 8.19, Lemma 8.7, and Lemma 8.17, the proof is clear. □

Lemma 8.20 shows that normal ∇-spectral spaces are uniquely determined with pairs (X, π), where X is a spectral space and $\pi : X \to X$ is a spectral map such that $\downarrow \pi[Y]$ is doubly spectral subset, for any doubly spectral subset Y. Note that this characterization of normal ∇-spectral spaces shows that ∇-spectral spaces can be considered as some sort of dynamic spectral spaces. To complete that topological picture, Lemma 8.21 connects full and faithful ∇-spaces with pairs (X, π), where π is a regular embedding and an epic map in **Spec**, respectively.

Theorem 8.22. *(Spectral duality for distributive ∇-algebras) For any $C \subseteq \{N, H, R, L, Fa, Fu\}$, we have $\mathbf{Alg}_\nabla(D, C) \simeq \mathbf{Spec}^{op}_\nabla(C)$, via the functors* $\bar{\mathbf{S}}$

and $\bar{\mathbf{A}}$ *and the natural isomorphisms* α *and* β *inherited from the bridge in Theorem 8.12. The same also holds for* $\mathbf{Alg}^H_\nabla(D, C)$ *and* $[\mathbf{Spec}^H_\nabla(C)]^{op}$.

Proof. Using Lemma 8.19 and the duality Theorem 8.10, the claim easily follows. □

9 Ring-theoretic Representations

Let R be a commutative unital ring. By a radical I of R, we mean a subset of R such that $x - y \in I$ and $rx \in I$, for any $x, y \in I$ and $r \in R$. For any set $A \subseteq R$, by $\langle A \rangle$, we mean the least ideal extending $A \subseteq R$. An ideal is called finitely generated if there exists a finite set A such that $I = \langle A \rangle$. An ideal is called prime if $xy \in I$ implies either $x \in I$ or $y \in I$. The radical of an ideal I is defined as $Rad(I) = \{x \in R \mid \exists n \in \mathbb{N}\ x^n \in I\}$. An ideal is called radical iff $Rad(I) = I$ and radically finitely generated if there exists a finitely generated ideal J such that $I = Rad(J)$. By $Spec(R)$, we mean the topological space of all prime ideals of R with the topology $\{U_r \mid r \in R\}$ where $U_r = \{P \in Spec(R) \mid r \notin P\}$. The following facts are all well-known: ([])

- The space $Spec(R)$ is spectral and for any ring homomorphism $f : R \to S$, the induced map $Spec(f) = f^{-1} : Spec(S) \to Spec(R)$ is a spectral map.

- (Hochster's Theorem) Conversely, for any spectral spaces X and Y such that $X \neq Y$ and any spectral map $F : X \to Y$, there exist commutative unital rings R_X and R_Y, a ring homomorphism $f : R_Y \to R_X$ and homeomorphisms $\alpha_X : Spec(R_X) \simeq X$ and $\alpha_Y : Spec(R_Y) \simeq Y$ such that $F \circ \alpha_X = \alpha_Y \circ Spec(f)$.

- The poset of all radical ideals of R, denoted by $\mathcal{RI}(R)$, is a locale and its Heyting implication is $[J : I] = \{x \in R \mid \forall y \in I\ xy \in J\}$. Note that if J is a radical ideal then so is $[J : I]$.

- Define $\mathcal{I} : \mathcal{O}(Spec(R)) \to \mathcal{RI}(R)$ by $\mathcal{I}(U) = \{r \in R \mid \forall P \notin U\ r \in P\}$ and $\mathcal{U} : \mathcal{RI}(R) \to \mathcal{O}(Spec(R))$ by $\mathcal{U}(I) = \{P \in Spec(R) \mid \exists x \in I\ x \notin P\}$. Then $\mathcal{I}$ and $\mathcal{U}$ establish a localic isomorphism between $\mathcal{O}(Spec(R))$ and $\mathcal{RI}(R)$.

- Under $\mathcal{I} - \mathcal{U}$ correspondence, the compact elements of $\mathcal{O}(Spec(R))$ correspond to the radically finitely generated ideals of R. The set of these ideals is denoted by $\mathcal{RI}_f(R)$.

- For any ring homomorphism $f : R \to S$, the map $f_* : \mathcal{RI}(R) \to \mathcal{RI}(S)$ defined by $f_*(I) = Rad(\langle f[I] \rangle)$ is a localic map and it is the $\mathcal{I} - \mathcal{U}$ counterpart of $Spec(f)^{-1} : \mathcal{O}(Spec(R)) \to \mathcal{O}(Spec(S))$, i.e., $f_* = \mathcal{I} \circ Spec(f)^{-1} \circ \mathcal{U}$. Moreover, $f_* \dashv f^{-1}$.

- If $F : Spec(R) \to Spec(S)$ is a continuous map, we have the localic map $F^{-1} : \mathcal{O}(Spec(S)) \to \mathcal{O}(Spec(R))$. Then, the $\mathcal{I} - \mathcal{U}$ correspondence assigns the correspondent localic map $\hat{F} : \mathcal{RI}(S) \to \mathcal{RI}(R)$, defined by $\hat{F} = \mathcal{I} \circ F^{-1} \circ \mathcal{U}$. This assignment is functorial, i.e., $\hat{id} = id$ and $\widehat{GH} = \hat{G}\hat{H}$, for any two continuous maps G, H.

- For any ring homomorphism $f : R \to S$, the map $Spec(f) : Spec(S) \to Spec(R)$ is surjective iff for any $I \in \mathcal{RI}(R)$, we have $\pi^{-1}(\pi_*(I)) = I$. It is a topological embedding iff for any $s \in S$, there exist a natural element n, an element $r \in R$ and a unit $u \in S$ such that $s^n = \pi(r)u$.

One immediate consequence of the previously mentioned connection is the following representation theorem:

Theorem 9.1. *(Ring-theoretic representation of Heyting algebras) For any Heyting algebra $\mathcal{H}$, there exists a commutative unital ring R such that $\mathcal{H}$ is isomorphic to the lattice of the radically finitely generated ideals of R.*

Proof. First, note that any Heyting algebra is a Heyting ∇-algebra that is both left and right. Then, by Theorem 8.22, there is a Heyting ∇-spectral space (X, R) such that the ∇-algebra of the compact opens of (X, R) is isomorphic to $\mathcal{H}$. Note that the bounded lattice embedding $i_{\mathcal{H}} : \mathcal{H} \to CO(X)$ preserves the Heyting implication, where $CO(X)$ is the lattice of compact opens of X. Now, by Hochster's Theorem, there exists a commutative unital ring R such that X is homeomorphic to $Spec(R)$. Therefore, by the properties mentioned above, $\mathcal{RI}(R) \simeq \mathcal{O}(X)$ as locales. More specifically, the compact opens of the space X correspond to the radically finitely generated ideals of R which implies that $\mathcal{H}$ is isomorphic to the lattice of the radically finitely generated ideals of R. □

Corollary 9.2. IPC *is sound and complete with respect to its algebraic interpretation in the locale of the radical ideals of commutative unital rings. Even more uniformly, there exists a commutative unital ring R such that the set of its radically finitely generated ideals, $\mathcal{RI}_f(R)$, is closed under operation $[J : I]$ and* **IPC** *is sound and complete with respect to its algebraic interpretation in the Heyting algebra $(\mathcal{RI}_f(R), \subseteq)$.*

Theorem 9.3. *Let R and S be commutative unital rings, $\pi : R \to S$ be a ring homomorphism and $f : Spec(R) \simeq Spec(S)$ be a homeomorphism. Then the tuple $(\mathcal{RI}(R), \nabla, \to)$ is a normal ∇-algebra, where*

$$\nabla I = \hat{f}(\pi_*(I)) \quad \textit{and} \quad I \to J = \pi^{-1}(\hat{g}[J : I])$$

and g is the inverse of f. Moreover, if $\to$ maps $\mathcal{RI}_f(R)$ to itself, $\mathcal{RI}_f(R)$ is also a normal ∇-algebra. Finally,

- *if $\pi^{-1}(\pi_*(I)) = I$, for any $I \in \mathcal{RI}(R)$, then, the normal ∇-algebra $(\mathcal{RI}(R), \nabla, \to)$ is faithful.*
- *if for any $s \in S$, there exist a natural element n, an element $r \in R$ and a unit $u \in S$ such that $s^n = \pi(r)u$, then, the normal ∇-algebra $(\mathcal{RI}(R), \nabla, \to)$ is full.*

Proof. First, note that the operations ∇ and $\to$ are well-defined. The case for ∇ is clear. For $\to$, note that if I and J are radical ideals of R, then so is $[J : I]$ and hence $\hat{g}([J : I])$ is a radical ideal of S. Finally, since π^{-1} maps radical ideals of S to the radical ideals of R, the set $I \to J$ is a radical ideal of R. For the adjunction condition, note that

$$\hat{f}(\pi_*(I)) \cap J \subseteq K \quad \text{iff} \quad I \subseteq \pi^{-1}(\hat{g}[K : J])$$

trivially holds, simply because $\hat{f}\hat{g} = \hat{g}\hat{f} = id$ and $\pi_* \dashv \pi^{-1}$. For normality, first note that $1 = \pi(1) \in \pi[R]$ and hence $\pi_*(R) = S$ which implies that $\hat{f}\pi_*(R) = R$, since $\hat{f}$ preserves the top element of the lattice. Secondly, we have to show that $\pi_*(I \cap J)) = \pi_*(I) \cap \pi_*(J)$. One direction is easy. For the other direction, if $x \in \pi_*(I) \cap \pi_*(J)$, there are natural numbers m and n such that $x^n \in \langle \pi[I] \rangle$ and $x^m \in \langle \pi[J] \rangle$. Therefore, $x^{m+n} \in \langle \pi[I \cap J] \rangle$ which implies $x \in \pi_*(I \cap J)$. Now, since $\hat{f}$ is a localic isomorphism, $\nabla = \hat{f}\pi_*$ must preserves finite intersections. Finally, note that if I is radically finitely generated, then so is $\hat{f}(\pi_*(I))$, because if $I = Rad(\langle A \rangle)$, for a finite set A, since $\pi_*(I) = Rad(\langle \pi[A] \rangle)$ and $\pi[A]$ is trivially finite, the ideal $\pi_*(I)$ is radically finitely generated. Since $\hat{f}$ is a localic isomorphism and hence preserves the compact elements of the locale, namely the radically finitely generated ideals, $\hat{f}(\pi_*(I))$ is also radically finitely generated. Therefore, if $\to$ maps $\mathcal{RI}_f(R)$ to itself, $\mathcal{RI}_f(R)$ is closed under ∇ and $\to$. This makes the set of all radically finitely generated ideals of R a normal ∇-algebra itself. Note that $\mathcal{RI}_f(R)$ is closed under all finite joins and meets of $\mathcal{RI}(R)$. Finally, the cases for faithfulness and fullnes are easy consequences of the characterization of a homomorphism π whose $Spec(\pi)$ is surjective or a topological embedding as we had in the beginning of this section together with Lemma 8.21. □

Definition 9.4. Let R and S be commutative unital rings, $\pi : R \to S$ be a ring homomorphism and $f : Spec(R) \simeq Spec(S)$ be a homeomorphism. Then the normal ∇-algebra $(\mathcal{RI}(R), \nabla, \to)$ as defined in Theorem 9.3 is called a spectral ∇-algebra. A spectral ∇-algebra is called closed if $\to$ maps $\mathcal{RI}_f(R)$ to itself. It is called faithful if $\pi^{-1}(\pi_*(I)) = I$, for any $I \in \mathcal{RI}(R)$ and full if for any $s \in S$, there exist a natural element n, an element $r \in R$ and a unit $u \in S$ such that $s^n = \pi(r)u$.

Theorem 9.5. *(Ring-theoretic representation) Let $C \in \{Fa, Fu\}$. For any $\mathcal{A} \in \mathbf{Alg}_\nabla(N, C)$, there exists a closed spectral ∇-algebra $(\mathcal{RI}(R), \nabla, \to) \in \mathbf{Alg}_\nabla(N, C)$ such that $\mathcal{A}$ is isomorphic to the ∇-algebra of radically finitely-generated ideals of $(\mathcal{RI}(R), \nabla, \to)$. Specially, any normal ∇-algebra is a subalgebra of a spectral ∇-algebra.*

Proof. First, let us make a convention. Since in any ∇-algebra, the operation $\to$ is uniquely determined by ∇, for simplicity, throughout this proof, we will denote a ∇-algebra only by its base lattice and its ∇ operator. Suppose $\mathcal{A}$ is a normal ∇-algebra. Then, by Theorem 8.22, there is a normal ∇-spectral space (X, R) such that the ∇-algebra of the compact opens of (X, R) is isomorphic to $\mathcal{A}$. Since (X, R) is normal, there is a function $h : X \to X$ such that $(x, y) \in R$ iff $x \in Cl(\{h(y)\})$. By Lemma 8.20, h is a spectral map. Now, let Y be a space homeomorphic to X but $X \neq Y$. Call the homeomorphism $g : Y \to X$. This is just a technical condition to make the Hochster's Theorem applicable. Since $h : X \to X$ is spectral and g is a homeomorphism, $hg : Y \to X$ is also spectral. By Hochster's Theorem, there exist rings R and S, a ring homomorphism $\pi : R \to S$ and homeomorphisms $\alpha_X : Spec(R) \simeq X$ and $\alpha_Y : Spec(S) \simeq Y$ such that $(hg) \circ \alpha_Y = \alpha_X \circ Spec(\pi)$:

$$\begin{array}{ccc} Spec(S) & \xrightarrow{\alpha_Y} & Y \\ {\scriptstyle Spec(\pi)}\downarrow & & \downarrow{\scriptstyle hg} \\ Spec(R) & \xrightarrow[\alpha_X]{} & X \end{array}$$

Define $f : Spec(R) \to Spec(S)$ as $f = \alpha_Y^{-1} \circ g^{-1} \circ \alpha_X$. This map is clearly a homeomorphism. Therefore, considering the rings R, S and maps $\pi : R \to S$ and $f : Spec(R) \to Spec(S)$, we will have a spectral ∇-algebra $(\mathcal{RI}(R), \hat{f}\pi_*)$. We claim that this spectral ∇-algebra works. To prove that, note that using the $\mathcal{I} - \mathcal{U}$ correspondence, this ∇-algebra is isomorphic to the ∇-algebra $(\mathcal{O}(Spec(R)), f^{-1} \circ Spec(\pi)^{-1})$. Since the isomorphism is also a locale isomorphism, the compact open elements of two sides are related by the isomorphism. Hence, it is enough to show that the ∇-algebra $(\mathcal{O}(X), h^{-1})$ is isomorphic to $(\mathcal{O}(Spec(R)), f^{-1} \circ Spec^{-1}(\pi))$ in a way that the isomorphism induces a locale isomorphism between the underline locales. For this isomorphism use the $\alpha_X^{-1} : \mathcal{O}(X) \to \mathcal{O}(Spec(R))$ and note that α_X^{-1} respects the ∇ operators of $(\mathcal{O}(Spec(R)), f^{-1} \circ Spec^{-1}(\pi))$ and $(\mathcal{O}(X), h^{-1})$, because

$$f^{-1} \circ Spec(\pi)^{-1} \alpha_X^{-1} = \alpha_X^{-1} g \alpha_Y Spec(\pi)^{-1} \alpha_X^{-1} = \alpha_X^{-1} g \alpha_Y \alpha_Y^{-1} g^{-1} h^{-1} = \alpha_X^{-1} h^{-1}.$$

Hence, the locale isomorphism α_X^{-1} preserves the ∇'s

$$\begin{array}{ccc} \mathcal{O}(X) & \xrightarrow{\alpha_X^{-1}} & \mathcal{O}(Spec(R)) \\ \downarrow & & \downarrow {\scriptstyle f^{-1} \circ Spec(\pi)^{-1}} \\ \mathcal{O}(X) & \xrightarrow[\alpha_X^{-1}]{} & \mathcal{O}(Spec(S)) \end{array}$$

and hence the implications. Therefore, $\mathcal{A}$ is isomorphic to the ∇-algebra of compact elements of the ∇-algebra $(\mathcal{RI}(R), \hat{f}\pi_*)$, which is the set of all radically finitely generated ideals. The only thing remained to prove is the conditions in $\{Fa, Fu\}$. We prove the faithfulness. Fullness is similar. If $\mathcal{A}$ is faithful, then by Theorem 3.4, we know that (X, R) is faithful. Then, by Lemma 8.17, h must be surjective. Since g is a homeomorphism, hg is also surjective which implies that $Spec(\pi)$ is surjective. Finally, this implies that $\pi^{-1}(\pi_*(I)) = I$, for any $I \in \mathcal{RI}(R)$. □

Corollary 9.6. *Let $C \in \{Fa, Fu\}$. The logic* $\mathbf{STL}(N, C)$ *is sound and complete with respect to its algebraic interpretation in ∇-spectral spaces, satisfying the conditions in C.*

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