# Generalized Heyting Algebras and Duality

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- For *bounded lattices*, set $\nabla c = 0$ and $a \to b = 1$, for any $a, b, c \in A$.

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- For *Heyting algebras*, set $\nabla c = c$ and $a \to b = a \supset b$, for any $a, b, c \in A$, where $\supset$ is the Heyting implication.

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Hence, $\nabla$-algebras generalize both bounded lattices and Heyting algebras.

# Intuitionistic Kripke Frames

## Example

Let $(W, \leq)$ be a poset. By an *intuitionistic Kripke frame*, we mean a tuple $\mathcal{K} = (W, \leq, R)$, where $R$ is a binary relation over $W$, compatible with the partial order, i.e., if $k' \leq k \, R \, l \leq l'$, then $k' \, R \, l'$, for any $k, k', l, l' \in W$. To any intuitionistic Kripke frame, we can assign a canonical $\nabla$-algebra, encoding its structure via topology. Set $\mathcal{X}$ as the locale of all upsets of $(W, \leq)$ and define $\nabla : \mathcal{X} \to \mathcal{X}$ as $\nabla_{\mathcal{K}} U = \{x \in W \mid \exists y \in U \; R(y, x)\}$ and $U \to_{\mathcal{K}} V = \{x \in W \mid \forall y \in W[R(x, y) \wedge y \in U \Rightarrow y \in V]\}$. It is easy to see that $(\mathcal{X}, \nabla_{\mathcal{K}}, \to_{\mathcal{K}})$ is a $\nabla$-algebra.

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## Motivation I: Intuitionistic Temporal Logics

A Heyting $\nabla$-algebra is the algebraic model for basic intuitionistic temporal logic.

# Dynamic systems

## Example

Let $X$ be a topological space and $f : X \to X$ be a continuous function. Define $\to_f$ over $\mathcal{O}(X)$ by $U \to_f V = f_*(int[U^c \cup V])$, where $f_* : \mathcal{O}(X) \to \mathcal{O}(X)$ is the right adjoint of $f^{-1}$. Then, the structure $(\mathcal{O}(X), f^{-1}, \to_f)$ is a $\nabla$-algebra. This $\nabla$-algebra is the point-free version of the dynamic system $(X, f)$, using the adjunction $f^{-1} \dashv f_*$ to encode the map $f$.

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**Motivation II: Point-free Dynamic Systems**

A Heyting $\nabla$-algebra in which $\nabla$ commutes with all finite meets is the elementary and point-free version of dynamic systems.

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## Motivation II: Point-free Dynamic Systems

A Heyting $\nabla$-algebra in which $\nabla$ commutes with all finite meets is the elementary and point-free version of dynamic systems.

## Motivation III: Implications

$\nabla$-algebras represent all possible implications...

# Varieties of $\nabla$-algebras

A $\nabla$-algebra is called:

- **(D)**: distributive, if $\mathcal{A}$ is distributive.
- **(N)**: normal, if $\nabla$ commutes with all finite meets.
- **(Fa)**: faithful, if $\nabla$ is surjective.
- **(Fu)**: full, if $\Box$ is surjective, where $\Box a = 1 \to a$.

For any $C \subseteq \{D, N, Fa, Fu\}$, by $\mathcal{V}(C)$ we mean the class of all $\nabla$-algebras with the properties described in the set $C$.

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The last three has a topological root. More precisely, over locales and in the presence of enough separation axioms on the space:

- **(N)**: $\nabla$ is the inverse image of a continuous function.
- **(N)+(Fa)**: The continuous function is a topological embedding.
- **(N)+(Fu)**: The continuous function is surjective.

# Some Closure Properties

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**Theorem**

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*Let $C \subseteq \{N, Fa, Fu\}$. Then, the variety $\mathcal{V}(C)$ is closed under Dedekind-MacNeille completion.*

**Theorem (Amalgamation)**

The varieties $\mathcal{V}(D, N)$ and $\mathcal{V}(D, N, Fa)$ have the amalgamation property.

# Subdirectly Irreducible Normal Distributive $\nabla$-algebras

To investigate the structure of the varieties, we have to study the building blocks of the varieties:

**Theorem**

*A non-trivial normal distributive $\nabla$-algebra $\mathcal{A}$ is subdirectly irreducible iff there exists $x \in A - \{1\}$ such that for any $y \in A - \{1\}$, there exist $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i} \Box^{n_i} y \leq x$.*

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**Example**

For Heyting algebras, where $\nabla a = \Box a = a$, the theorem states that $\mathcal{A}$ is subdirectly irreducible iff there exists $x \in A - \{1\}$ such that $y \leq x$, for any $y \in A - \{1\}$. This means that $\mathcal{A}$ has the second greatest element.

# Simple Normal Distributive $\nabla$-algebras

**Theorem**

*A normal distributive $\nabla$-algebra $\mathcal{A}$ is simple iff for any $x \in A - \{1\}$, there exist $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i} \Box^{n_i} x = 0$.*

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For Heyting algebras, where $\nabla a = \Box a = a$, the theorem states that $\mathcal{A}$ is simple iff for any $x \in A - \{1\}$ we have $x = 0$. This means that $\mathcal{A}$ is the boolean algebra $\{0, 1\}$.

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**Theorem**

*There are infinitely many simple finite normal Heyting $\nabla$-algebras.*

# Variants of Kripke Frames

Variants of Kripke frames are defined by:

**Definition**

Let $(W, \leq, R)$ be an intuitionistic Kripke frame:

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- **(Fa)**: If for any $x \in W$, there exists $y \in W$ such that $(y, x) \in R$ and for any $z \in W$ such that $(y, z) \in R$ we have $x \leq z$.
- **(Fu)**: If for any $x \in W$, there exists $y \in W$ such that $(x, y) \in R$ and for any $z \in W$ such that $(z, y) \in R$ we have $z \leq x$.

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For any $C \subseteq \{N, Fa, Fu\}$, by $\mathbf{K}(C)$, we mean the class of all Kripke frames with the properties described in the set $C$.

- **(N)+(Fa)**: The function $\pi$ is an order embedding.
- **(N)+(Fu)**: The function $\pi$ is surjective.

# A Representation Theorem

For any $C \subseteq \{N, Fa, Fu\}$, if a Kripke frame is in $\mathbf{K}(C)$, then its corresponding $\nabla$-algebra is in $\mathcal{V}(C, D)$.

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For any $C \subseteq \{N, Fa, Fu\}$ and any $\mathcal{A} \in \mathcal{V}(C, D)$, there is a Kripke frame $\mathcal{K} \in \mathbf{K}(C)$ and a $\nabla$-algebra embedding from $\mathcal{A}$ into the $\nabla$-algebra corresponding to $\mathcal{K}$.

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Is it possible to strengthen this representation theorem to a full scale duality theory?

# $\nabla$-spaces as the Unification of Priestley and Esakia Spaces

## Definition

A $\nabla$-space is a tuple $(X, \leq, R)$ of a Priestley space $(X, \leq)$ and a binary relation $R$ on $X$ such that:

- $R$ is compatible with the order, i.e., if $x' \leq x$, $(x, y) \in R$ and $y \leq y'$, then $(x', y') \in R$,
- $R[x] = \{y \in X \mid (x, y) \in R\}$ is closed, for every $x \in X$,
- $\Diamond_R(U) = \{x \in X \mid \exists y \in U\ (x, y) \in R\}$ is clopen, for any clopen $U$,
- $\nabla_R(V) = \{x \in X \mid \exists y \in V\ (y, x) \in R\}$ is a clopen upset, for any clopen upset $V$.

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- For $R = \emptyset$, a $\nabla$-space is just a *Priestley* space.
- For $R = \leq$, a $\nabla$-space is just an *Esakia* space.

# Variants of $\nabla$-spaces

Note that any $\nabla$-space is a Kripke frame, if we forget the topology of the space. A $\nabla$-space satisfies a condition in the set $\{N, Fa, Fu\}$, if it satisfies the condition as a Kripke frame.

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- **(N)**: The function $\pi : X \to X$ is a Priestley map and $\downarrow \pi[U]$ is clopen, for any clopen $U$.
- **(N)+(Fa)**: The function $\pi$ is also an order embedding or a regular monic in the category of Priestley spaces.
- **(N)+(Fu)**: The function $\pi$ is also surjective or an epic map in the category of Priestley spaces.

## Definition

By a $\nabla$-space map $f : (X, \leq_X, R_X) \to (Y, \leq_Y, R_Y)$, we mean an order-preserving continuous map such that:

- For any $x, x' \in X$, if $(x, x') \in R_X$ then $(f(x), f(x')) \in R_Y$,
- for any $y' \in Y$ such that $(f(x), y) \in R_Y$, there exists $x' \in X$ such that $(x, x') \in R_X$ and $f(x') = y$,
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Let $C \subseteq \{N, Fa, Fu\}$. The $\nabla$-spaces in $\mathbf{K}(C)$ together with $\nabla$-space maps form a category. Denote this category by $\mathbf{Space}_\nabla(C)$. If we also denote the category of all $\nabla$-algebras in $\mathcal{V}(D, C)$ together with corresponding algebraic morphisms by $\mathbf{Alg}_\nabla(D, C)$, then:

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### Theorem *(Priestley-Esakia duality for distributive $\nabla$-algebras)*

Let $C \subseteq \{N, Fa, Fu\}$. Then, $\mathbf{Alg}_\nabla(D, C) \simeq \mathbf{Space}^{op}_\nabla(C)$.

Thank you for your attention!