Intuitionistic Implications: On the Logics of Spacetime

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Our Weekly Online Meeting, Utrecht, Pandemic 2020

The Episodes

- **The Algebro-Topological Episode:** What is an implication? What are their characterizations? How they relate to the philosophy of intuitionism?

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- **The Logical Episode:** What is the logic of implication? What are the well-behaved conservative extensions of this logic? What is the relationship between the implications and the usual intuitionistic implication? What is the proof theory of the implication?

Episode I: Examples of Implications

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The Main Problem

What is the abstract and the most general notion of implication?

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- Transitivity, i.e., "$A \vdash B$ *and* $B \vdash C$ *implies* $A \vdash C$" for any propositions A, B, and C. The internalization:

  $$(A \to B) \wedge (B \to C) \vdash (A \to C),$$

  for any propositions A, B, and C.

Abstract Implication

Definition

Let $\mathcal{A} = (A, \leq, \wedge, 1)$ be a bounded meet-semilatice. By an implication $\rightarrow: A^{op} \times A \Rightarrow A$ we mean any monotone function with the following properties:

- $a \rightarrow a = 1$,
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The structure $(A, \leq, \wedge, 1, \rightarrow)$ is called a strong algebra if $\rightarrow$ is an implication.

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- Let $\mathcal{A}$ be a bounded meet-semilatice. Define $a \rightarrow b = 1$ for all $a, b \in A$.
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- Gödel's implication on $[0, 1]$ defined by $a \to b = b$ if $a > b$ and 1 otherwise.

Two Construction Methods

- Let $(A, \leq, \wedge, 1, \to)$ be a strong algebra and $F : A \to A$ be a monotone operation. Define $a \to_F b = F(a) \to F(b)$. Then $\to_F$ is also an implication.

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These two methods, applied on the intuionistic implication (on $\mathcal{O}(X)$), construct all possible implications.

The first method is the modification factor. However, the applications of the second method on the intuionistic implications play a critical philosophical role. We call these implications *generalized intuitionistic implications.*

Intuitionism: Propositions via Space

Let S be the set of all creative subject's mental states. Then by a proposition P we mean a subset of S consisting of all states in which P holds and this fact is verifiable by finite means. *Finiteness* imposes two conditions:

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- *Finite Intersection.* If both A and B are finitely verifiable propositions, then so is $A \wedge B$. Because, if $A \wedge B$ holds in a state, there are finite verifications for both of them and the combination of these verifications is also finite. Note that the same claim is not necessarily true for infinite conjunctions, because, if the infinite conjunction is true, we need possibly infinite number of verifications that may exceed any possible finite memory.

- *Arbitrary Union.* For some set I, if A_i is finitely verifiable for any $i \in I$, then so is $\bigvee_{i \in I} A_i$. Because, if $\bigvee_{i \in I} A_i$ holds in a state, then one of them must hold and since it has a finite verification, the verification also works for the whole disjunction.

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Therefore, it should not be surprising that intuitionistic propositional logic is sound and complete with respect to its topological interpretation that reads a proposition as an open subset of a given topological space. In this sense, intuitionism may be interpreted as the logic of space as opposed to the classical logic that corresponds to the logic of sets or discrete spaces. Compare the set of all opens of a space to the opens of a discrete space, namely the Boolean algebra of all subsets.

Intuitionism: The Temporal Structure

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- ∇A is a proposition itself. Since, if ∇A holds in a mental state, there is some point in the past in which A holds. But A is a proposition and hence has a finite verification at that point. Therefore, it is easy to bring that verification to the current mental state and save it as some temporal information of the past.

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- ∇ is clearly monotone and union preserving. If $\nabla \bigvee_{i \in I} A_i$ holds at some state, then there *exists* some point in the past in which $\bigvee_{i \in I} A_i$ holds. Hence, one of A_i's must hold in that point which implies ∇A_i holds at the current state. Hence, we have $\bigvee_{i \in I} \nabla A_i$.

Spacetimes

The spatio-temporal structure of the creative subject's mental states is formalized by:

Definition

Let X be a topological space and $\nabla : \mathcal{O}(X) \to \mathcal{O}(X)$ be an increasing and join preserving operation. Then the pair (X, ∇) is called a spacetime.

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Example

Let $\mathcal{K} = (K, \leq, R)$ be an intuitionistic Kripke model. Then the pair $(UP(K, \leq), \nabla_{\mathcal{K}})$ is a spacetime, where $UP(K, \leq)$ is the upset space $(K, \leq)$ and $\nabla_{\mathcal{K}}(U) = \{x \in K | \exists y \in U \text{ such that } (y, x) \in R\}$.

Generalized Intuitionistic Implications

Theorem

Let (X, ∇) be a spacetime. Then there exists an implication $\to_\nabla$ on $\mathcal{O}(X)$ called generalized intuitionistic implication such that for any $U, V, W \in \mathcal{O}(X)$ we have $\nabla W \cap U \subseteq V$ iff $W \subseteq U \to_\nabla V$, i.e., $\nabla(U \to_\nabla V) \cap U \subseteq V$ and $U \to_\nabla V$ is the best such proposition.

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Proof.

Define $G(U) = \bigcup\{V | \nabla V \subseteq U\}$ and $U \to_\nabla V$ as $G(int(U^c \cup V))$. It is easy to show that G is meet-preserving. One side of the equivalence is obvious. The other side is the result of join preservability of ∇. Note that $\to_\nabla$ is the result of the application of the second method on intuitionistic implication on $\mathcal{O}(X)$.

Representation Theorems I

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General Representation Theorem

If $\mathcal{A}$ is a strong algebra then there exists a spacetime (X, ∇) and a meet semi-lattice embedding $i : A \to \mathcal{O}(X)$ and a monotone map $F : \mathcal{O}(X) \to \mathcal{O}(X)$ such that for any $a, b \in A$ we have $i(a \to b) = F(i(a)) \to_{\nabla} F(i(b))$.

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Philosophical Consequence

Any implication is a *generalized intuitionistic implication* up to a modification factor and enlarging the domain of the discourse.

Meet-internalizing Implications

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This is impossible. The reason is that for any spacetime (X, ∇), the implication $\to_\nabla$ has the following *meet-internalizing* property:

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"*U implies (V and W) iff [U implies W] and [U implies W]*."

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because,

$$\nabla Z \cap U \subseteq V \cap W \;\text{ iff }\; Z \subseteq U \to_\nabla V \cap W$$

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Representation Theorems II

Therefore, the necessary condition for an abstract implication to be embeddable in a spacetime is the meet-internalizing condition. This condition is fortunately sufficient:

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Special Representation Theorem (A., Alizadeh, Memarzadeh)

If $\mathcal{A}$ is a meet internalizing strong algebra, then there exists a spacetime (X, ∇) and a strong algebra embedding $i : \mathcal{A} \to (\mathcal{O}(X), \to_\nabla)$.

Representation Theorems II

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Philosophical Consequence

Any *reasonable* implication is a *generalized intuitionistic implication*, enlarging the domain of the discourse.

Episode II: Implicational Systems

The weak implicational systems are usually defined as an extension of the system **F** defined as the system including the axioms, the conjunction and the disjunction rules of **LJ** plus the following four rules:

$$\frac{\Gamma \Rightarrow A \to B \qquad \Gamma \Rightarrow B \to C}{\Gamma \Rightarrow A \to C} \qquad \frac{A \Rightarrow B}{\Rightarrow A \to B}$$

$$\frac{\Gamma \Rightarrow A \to B \qquad \Gamma \Rightarrow A \to C}{\Gamma \Rightarrow A \to (B \wedge C)} \qquad \frac{\Gamma \Rightarrow A \to C \qquad \Gamma \Rightarrow B \to C}{\Gamma \Rightarrow (A \vee B) \to C}$$

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It is also possible to add some additional rules to **F** such as:

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Theorem (Kripke Semantics)

The system **F** is sound and complete with respect to all Kripke models.

Spacetime Logics

Let $\mathcal{L}_\nabla$ be the usual language of propositional logic with a unary modal operator ∇. Define **STL** as the system consisting of the usual sequent-style rules for all connectives except implication (and hence negation) plus:

Implication Rules:

$$\frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, \nabla(A \to B) \Rightarrow C} \; L\to \qquad \frac{\nabla\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \; R\to$$

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Modal Rule:

$$\frac{\Gamma \Rightarrow A}{\nabla\Gamma \Rightarrow \nabla A}\, \nabla$$

Γ includes exactly one formula. If Γ can be arbitrary, the stronger rule is called (N) and the stronger system is $\mathbf{STL}(N)$.

Some Proof Trees in **STL**

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Topological Semantics

Definition

A topological model is a tuple (X, ∇, V) such that (X, ∇) is a spacetime and $V : \mathcal{L}_\nabla \to \mathcal{O}(X)$ is a valuation function such that: $V(\top) = X$; $V(\bot) = \emptyset$; $V(A \wedge B) = V(A) \cap V(B)$; $V(A \vee B) = V(A) \cup V(B)$; $V(A \to B) = V(A) \to_\nabla V(B)$ and $V(\nabla A) = \nabla V(A)$. We say $(X, \nabla, V) \vDash \Gamma \Rightarrow A$ when $\bigcap_{\gamma \in \Gamma} V(\gamma) \subseteq V(A)$.

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Soundness-completeness Theorem

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$\Gamma \vdash_{\mathbf{STL}} A$ iff $\Gamma \Rightarrow A$ is valid in all spacetimes.

Strong Completeness Theorem

For completeness any fixed discrete space with the cardinality greater than the continuum is sufficient.

An Embedding Theorem

What is the ∇-free fragment of the system **STL**?

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A ∇-free formula A is provable in **STL** iff it is provable in **F**. The system **F** is the propositional logic of all Kripke frames. (Not necessarily reflexive, transitive or persistent).

Proof.

We saw how to embed **F** into **STL**. For completeness, note that any Kripke model (K, R) can be seen as a discrete topological space with the union preserving operator ∇_R encoding the relational data R, where $\nabla_R(U) = \{x \in K | \exists y \in U \text{ such that } (y, x) \in R\}$. □

Kripke Models

Definition

- By a Kripke model for the language $\mathcal{L}_\nabla$, we mean an intuitionistic Kripke model i.e., a tuple $\mathcal{K} = (W, \leq, R, V)$ where $(W, \leq)$ is a poset, $R \subseteq W \times W$ is a relation over W (not necessarily transitive or reflexive) compatible with $\leq$, i.e., for all $u, u', v, v' \in W$ if $(u, v) \in R$ and $u' \leq u$ and $v \leq v'$ then $(u', v') \in R$ and $V : At(\mathcal{L}_\nabla) \to U((W, \leq))$ where $At(\mathcal{L}_\nabla)$ is the set of atomic formulas of $\mathcal{L}_\nabla$ and $U((W, \leq))$ is the set of all upsets of $(W, \leq)$.

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- Define the forcing relation as usual using the relation R and for the ∇ let $u \Vdash \nabla A$ if there exists $v \in W$ such that $(v, u) \in R$ and $v \Vdash A$.

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Theorem (Soundness-Completeness)

The logic **STL** is sound and complete with respect to all Kripke models.

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- $p^{\nabla} = \nabla \Box p$, $\bot^{\nabla} = \bot$ and $\top^{\nabla} = \top$.
- $(A \wedge B)^{\nabla} = A^{\nabla} \wedge B^{\nabla}$ and $(A \vee B)^{\nabla} = A^{\nabla} \vee B^{\nabla}$.
- $(A \to B)^{\nabla} = \nabla(A^{\nabla} \to B^{\nabla})$.

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Theorem

For any $\Gamma \cup A \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{IPC}} A$ *iff* $\Gamma^\nabla \vdash_{\mathbf{STL}(N)} A^\nabla$.

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For any $\Gamma \cup A \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{IPC}} A$ *iff* $\Gamma^\nabla \vdash_{\mathbf{STL}(N)} A^\nabla$.

This shows that the logic of spacetime is a refined version of the usual intuitionistic logic.

Proof Theory of Spacetime Logic (with A. Mahmoudian)

A sequent is the object $\langle \Gamma_n \rangle_{n=0}^{\infty} \Rightarrow \Delta$, where the left side is a sequence of multisets of formulas like:

$$\langle \Gamma_n \rangle_{n=0}^{\infty} = (\cdots | \Gamma_2 | \Gamma_1 | \Gamma_0)$$

where for except finitely many n's we have $\Gamma_n = \emptyset$. The interpretation of the sequence $\langle \Gamma_n \rangle_{n=0}^{\infty}$ is

$$\bigwedge_{n=0}^{\infty} (\bigwedge \nabla^n \Gamma_n),$$

where $\nabla^n \Pi = \{\nabla^n A | A \in \Pi\}$ in which ∇^n means n many ∇'s.

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Note that ϵ_n means $- | - | \cdots | -$ where the number of $|$'s are n and ϵ means the empty sequence $\langle \emptyset \rangle_{n=0}^{\infty}$. By $\cup$ we mean the pointwise union.

Consider the system $\mathbf{GSTL}(N)$ consisting of the following set of sequent-style rules:

Axioms:

$$\frac{}{\epsilon | A \Rightarrow A} \qquad \frac{}{\epsilon \Rightarrow \top} \qquad \frac{}{\epsilon | \bot, \epsilon_n \Rightarrow}$$

for any $n \geq 0$.

Structural Rules:

$$\frac{\mathcal{S}|\Gamma, \mathcal{T} \Rightarrow \Delta}{\mathcal{S}|\Gamma, A, \mathcal{T} \Rightarrow \Delta}\, Lw \qquad \frac{\mathcal{S} \Rightarrow}{\mathcal{S} \Rightarrow A}\, Rw \qquad \frac{\mathcal{S}|\Gamma, A, A, \mathcal{T} \Rightarrow \Delta}{\mathcal{S}|\Gamma, A, \mathcal{T} \Rightarrow \Delta}\, Lc$$

Cut:

$$\frac{\mathcal{S}|\Gamma, \mathcal{T} \Rightarrow A \qquad \mathcal{S}'|\Sigma, A, \mathcal{T}' \Rightarrow \Delta}{[\mathcal{S}'|\Sigma, \mathcal{T}'] \cup [\mathcal{S}|\Gamma, \mathcal{T}, \epsilon_n] \Rightarrow \Delta}\, cut$$

where n is the number of the symbol $|$ in $\mathcal{T}$.

Conjunction Rules:

$$\frac{\mathcal{S}|\Gamma, A, \mathcal{T} \Rightarrow \Delta}{\mathcal{S}|\Gamma, A \wedge B, \mathcal{T} \Rightarrow \Delta} L\wedge \quad \frac{\mathcal{S}|\Gamma, B, \mathcal{T} \Rightarrow \Delta}{\mathcal{S}|\Gamma, A \wedge B, \mathcal{T} \Rightarrow \Delta} L\wedge \quad \frac{\mathcal{S} \Rightarrow A \qquad \mathcal{S} \Rightarrow B}{\mathcal{S} \Rightarrow A \wedge B}$$

Disjunction Rules:

$$\frac{\mathcal{S}|\Gamma, A, \mathcal{T} \Rightarrow \Delta \qquad \mathcal{S}|\Gamma, B, \mathcal{T} \Rightarrow \Delta}{\mathcal{S}|\Gamma, A \vee B, \mathcal{T} \Rightarrow \Delta} L\vee \quad \frac{\mathcal{S} \Rightarrow A}{\mathcal{S} \Rightarrow A \vee B} R\vee \quad \frac{\mathcal{S} \Rightarrow B}{\mathcal{S} \Rightarrow A \vee B}$$

Modal Rules:

$$\frac{\mathcal{S}|\Gamma, \Sigma|\Pi, \mathcal{T} \Rightarrow \Delta}{\mathcal{S}|\Gamma|\nabla\Sigma, \Pi, \mathcal{T} \Rightarrow \Delta} L\nabla \qquad \frac{\mathcal{S} \Rightarrow \Delta}{\mathcal{S}|- \Rightarrow \nabla\Delta} R\nabla$$

Implication Rules:

$$\frac{\mathcal{S}|\Gamma|\Sigma, \mathcal{T} \Rightarrow \nabla^n A \qquad \mathcal{S}|\Gamma|\Sigma, B, \mathcal{T} \Rightarrow \Delta}{\mathcal{S}|\Gamma, A \to B|\Sigma, \mathcal{T} \Rightarrow \Delta} L\to \qquad \frac{\mathcal{S}|A \Rightarrow B}{\mathcal{S} \Rightarrow A \to B} R\to$$

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where n is the number of the symbol $|$ in $\mathcal{T}$.

Theorem

The system **GSTL**(N) is sound and complete for **STL**(N) = **STL** + N where N is commutativity of ∇ with all finite conjunctions.

Cut Elimination

The main feature of the system $\mathbf{GSTL}(N)$ is its cut elimination:

Theorem

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Corollary (Temporal Visser Rules)

The following rule is admissible in **STL**(**N**):

$$\frac{\{\nabla^{m_i+1}(A_i \to B_i)\}_{0 \leq i \leq n}, \{C_j \to D_j\}_{0 \leq i \leq m} \Rightarrow \nabla^{m_{n+1}} A_{n+1} \vee \nabla^{m_{n+2}} A_{n+2}}{\{\{\nabla^{m_i+1}(A_i \to B_i)\}_{0 \leq i \leq n}, \{C_j \to D_j\}_{0 \leq i \leq m} \Rightarrow \nabla^{m_i} A_i\}_{0 \leq i \leq n+2}}$$

Other Applications

As some more familiar applications, we have:

Corollary (Stronger Visser Rules)

The following rule is admissible in **STL**(**N**):

$$\frac{\{C_j \to D_j\}_{0 \leq i \leq m} \Rightarrow E \vee F}{\{C_j \to D_j\}_{0 \leq i \leq m} \Rightarrow E / \{C_j \to D_j\}_{0 \leq i \leq m} \Rightarrow F}$$

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Corollary (DP)

STL(**N**) has disjunction property.

Thank you for your attention!