Mining the Surface: The NP-Search Problems of Arithmetic

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Abstract

One of the elegant achievements in the history of proof theory is the characterization of the provably total recursive functions of the strong theories of arithmetic. This characterization relates the provability of the totality of a function on the one hand and its computability via the ordinal recursion on the theory's proof theoretic ordinal, on the other. Unfortunately, this correspondence is not sufficiently fine-grained to also understand the bounded world of the bounded low complexity functions, i.e., the bounded functions with low complexity definitions.

In this paper we intend to develop a refined version of the mentioned correspondence. We will characterize the provably total low complexity functions of a theory via a suitable family of syntactical ordinal-based algorithms. More precisely, we will show that a feasibly defined function is provably total in a theory iff there exists a sequence of PV-provable polynomial time reductions with the length of the proof theoretic ordinal of the theory, starting in a feasibly computable value and ending in the value of the function itself. This characterization is useful in the classification of the feasibly defined bounded functions in a theory. More specifically, it is useful in generalizing the Beckmann's characterization [1] of the total NP-search problems of PA to any theory with a reasonable ordinal analysis.

1 Introduction

One of the elegant achievements in the history of proof theory is the characterization of the provably total recursive functions of the different theories of arithmetic. This characterization employs the ordinal recursion on the proof theoretic ordinal of the theory, to first relate the provability and the growth rate and then to establish the independence of some Π_2^0 formulas from some strong arithmetical theories such as $I\Sigma_n$ and PA.

Trying to extend this characterization and hence its independence technique to some weaker fragments of formulas, one can observe that in the bounded world, the mentioned correspondence totally breaks down. First because in some formulas such as $\forall x A(x)$ where A is an atomic formula, there is no essentially existential quantifier to witness and secondly, even if we have a bounded existential quantifier in the formula, for instance in $\forall x \exists y \leq t B(x, y)$ where B is atomic, the characterization does not lead to a characterization of these functions withlow complexity definitions. The reason is the following: The best thing we can learn from the characterization is the existence of an ordinal recursive computation to find the bounded y. But this is much weaker than the provability of the formula $\forall x \exists y \leq t B(x, y)$ that we had. Because based on the recursive algorithm that we have, the function is represented by a formula of the form $\exists w C(x, w, y)$, where w encodes the ordinal recursive computation. But despite the bounded value of the function, this computation can be extremely huge and hence unbounded by the basic simple terms in the language. Hence, there is a difference between the representable bounded functions and the functions representable by bounded formulas.

In this paper we try to provide a generalization of this correspondence to also cover the low complexity statements. For this purpose, we develop a more faithful characterization of the provable formulas of the form $\forall x \exists y A(x, y)$ where A is a polynomial time computable predicate. This characterization provides a canonical decomposition of the proof to an ordinal-length PV-provable implications. It then leads to a representation for all universal formulas on the one hand and an algorithm to compute a witness for the existential formulas on the other. In the latter case, the algorithm reveals much more information than the previously known upper bounds. It replaces the growth rate of a function with a concrete algorithm for its computation to relate provability with a more fundamental notion of computational complexity. Therefore, this characterization will be useful in cases that the existential quantifier is bounded or even when we lack these quantifiers altogether.

Note that there is nothing specific to polynomial time predicates, the technique actually works for any strong enough complexity class and any theory for which we have a reasonable ordinal analysis. However, for simplicity we will only focus on polynomial time computation which makes our main result as the following:

Theorem 1.1. *Let $A(\vec{x}, y)$ be a polynomial time computable predicate and T an arithmetical theory with the proof theoretic ordinal α_T. Then $T \vdash \forall \vec{x} \exists y A(\vec{x}, y)$ iff there exists $\beta \prec \alpha_T$, a polynomial time computable predicate $G(\gamma, \vec{x}, \vec{z})$ and polynomial time computable functions $\vec{N}$, q and p such that:*

(*i*) $\mathrm{PV} \vdash G(\beta, \vec{x}, \vec{i}(\vec{x}))$,

(*ii*) $\mathrm{PV} \vdash \gamma \neq 0 \to q(\gamma, \vec{x}, \vec{z}) \prec \gamma$,

(*iii*) $\mathrm{PV} \vdash \gamma \neq 0 \to [G(\gamma, \vec{x}, \vec{z}) \to G(q(\gamma, \vec{x}, \vec{z}), \vec{x}, \vec{N}(\gamma, \vec{x}, \vec{z}))]$,

(*iv*) $\mathrm{PV} \vdash G(0, \vec{x}, \vec{z}) \to A(\vec{x}, p(\vec{x}, \vec{z}))$.

To prove this characterization, we will pursue the following strategy: First, in the next section, we will use a sequence of reductions mainly based on Mints' style technique of continuous cut elimination [5] to transfer the provable low complexity consequences of T to a simpler theory in the language of PV axiomatized by transfinite induction on the universally quantified polynomial time computable formulas. Then, we will continue the process of reductions to reduce the provability of these low complexity statements to a uniform ordinal-length sequence of PV-implications. The latter is called an ordinal flow and we will devote the last section to investigate its behaviour. Finally, using the witnessing in the theory PV we can prove the characterization that we have mentioned.

This characterization has its useful applications. For instance, it is useful to characterize the total NP-search problems of a theory for which we have a reasonable ordinal analysis. It generalizes the known characterization of the total NP-search problems of PA that is based on polynomial local search problems defined on ordinals less than ϵ_0. (See [1]). We will generalize this characterization to any theory with the proof theoretic ordinal α that has a polynomial time computable representation.

2 A Bridge of Reductions

Let us start with a theory T with a reasonable ordinal analysis. (The precise definition will be presented in a moment). And assume that the language of T has a function symbol for any polynomial time computable function (in the sense of PV) and T itself extends the theory PV. This condition makes it possible to talk about the low complexity predicates.

First let us define what we mean by a reasonable ordinal analysis of an arithmetical theory T. For this purpose we need the usual primitive recursive representation of the ordinals:

Definition 2.1. A primitive recursive representation of the ordinal α is a structure $\mathbb{A} = (A, \prec_A, +_A, \cdot_A, x \mapsto \omega^x)$ such that:

(i) A is an infinite primitive recursive subset of $\mathbb{N}$.

(ii) $\prec_A$ is a primitive recursive binary relation on A.

(iii) $+_A$, $\cdot_A$ are binary, and $x \mapsto \omega^x$ is unary, primitive recursive functions on A.

(iv) PRA proves that $\mathbb{A}$ satisfies *all the usual order and algebraic properties* of an initial segment of ordinals that are defined in detail in [5].

(v) The structure $\mathbb{A}$ is isomorphic to the structure $(\alpha, \prec_\alpha, +_\alpha, \cdot_\alpha, \beta \mapsto \omega^\beta)$ where the order and the functions in the latter structure are the usual ordinal theoretic order and the usual operations. Note that this condition implies that the ordinal α is closed under the operation $\beta \mapsto \omega^\beta$.

Remark 2.2. From now on, when we work with an ordinal α, we always fix a primitive recursive representation for it and later also a polynomial time computable representation. We use the lower Greek alphabets both for the ordinals and their numeral representations. For instance, by the arithmetical formula $H(\gamma, \vec{x})$ we mean $c \in A \wedge H(c, \vec{x})$ and by the quantifier $\forall \beta$ we actually mean $\forall b \in A$.

After arithmetizing the needed ordinal, we are ready to recall the definition of the Π^0_2-proof theoretical ordinal of the theory T:

Definition 2.3. Let T be a theory of arithmetic. We say that α is a Π^0_2-proof theoretical ordinal of T when $T \equiv_{\Pi^0_2} \mathrm{PA} + \bigcup_{\beta \prec \alpha} \mathrm{TI}(\prec_\beta)$, where $\mathrm{TI}(\prec_\beta)$ means the full transfinite induction up to the ordinal β:

$$\forall \gamma \prec \beta (\forall \delta \prec \gamma A(\delta) \to A(\gamma)) \to \forall \gamma \prec \beta A(\gamma)$$

The rest of this section is devoted to a sequence of reductions that transfer the low complexity consequences of the theory T to simpler more feasibly presented theories. This sequence consists of two steps. First we start with the following continuous cut elimination technique to reduce the full transfinite induction to the non-existence of decreasing primitive recursive sequences of ordinals. Then we continue with introducing a new totally feasibly represented system and interpreting this non-existence in that system:

Theorem 2.4. *[5] Let T be a theory of arithmetic and α its Π_2^0-proof theoretical ordinal. Then*

$$T \equiv_{\Pi_2^0} \mathrm{PRA} + \bigcup_{\beta \prec \alpha} \mathrm{PRWO}(\prec_\beta)$$

where $\mathrm{PRWO}(\prec_\beta)$ *is the scheme*

$$\forall \vec{x} \exists y [f(\vec{x}, y+1) \not\prec f(\vec{x}, y) \vee f(\vec{x}, y) \notin A \vee f(\vec{x}, y) \not\prec \beta]$$

for any function symbol f in the language of PRA.

For the second step, we need to improve our ordinal representation system from its primitive recursive setting to a polynomial time computable one:

Definition 2.5. Let α be an ordinal with a given primitive recursive representation. Then we say

$$\mathbb{A} = (A, \prec_A, +_A, \cdot_A, \dot{-}_A, d_A, o_A, \omega_A)$$

is a polynomial time representation of the ordinal α when A and $\prec_A$ are polynomial time relations, $+_A, \cdot_A, \dot{-}_A, d_A(\cdot,\cdot)$ and o_A are polynomial time functions and ω_A is a constant, such that:

(i) The structure $\mathbb{A} = (A, \prec_A, +_A, \cdot_A, \dot{-}_A, d_A, o_A, \omega_A)$ is isomorphic to the structure $(\alpha, \prec_\alpha, +_\alpha, \cdot_\alpha, \dot{-}_\alpha, d_\alpha, o_\alpha, \omega_\alpha)$ where $+_\alpha$, $\cdot_\alpha$ are the usual addition and product of ordinals and $\dot{-}_\alpha, d_\alpha$ are subtraction and division from left, i.e., for $\gamma \preceq \beta$ we have $\beta \dot{-}_\alpha \gamma = \delta$ where $\gamma +_\alpha \delta = \beta$ and otherwise, $\beta \dot{-}_\alpha \gamma = 0$. For division, if $\gamma \neq 0$, by $d_\alpha(\beta, \gamma)$ we mean the unique δ where $\beta = \gamma \cdot_\alpha \delta +_\alpha \lambda$ and $\lambda \prec \gamma$. Finally, o_α is a function that sends a natural number to the ordinal of that order type and ω_α is the ordinal ω.

(ii) PV proves the axioms of discrete ordered semi-rings for the structure $\mathbb{A}$ without the commutativity of addition and the axioms which state that $\prec_A$ is preserved under left addition and left multiplication by a non-zero element. And finally, o is an isomorphism between $(\mathbb{N}, <)$ and $(\{x \prec \omega_A\}, \prec_{\omega_A})$.

(ii) PRA proves that $(A, \prec_A)$ is equivalent to the primitive recursive representation of $(\alpha, \prec_\alpha)$ i.e., their *being an ordinal* predicates and their order predicates are equivalent over PRA.

Definition 2.6. The class $\forall_1$ of formulas is inductively defined as the least set of formulas in the language of PV that includes atomic formulas and is closed under conjunction, disjunction, implication with quantifier-free precedents and universal quantifiers.

And finally, we need a theory completely defined in the language of PV:

Definition 2.7. Let $\mathcal{L}_{\mathrm{PV}}$ be the language of PV. Define the system $\mathrm{TI}(\forall_1, \prec)$ as the theory PV together with the $\forall_1$-transfinite induction on $\prec$, i.e.,

$$\forall \delta (\forall \gamma \prec \delta \; A(\gamma) \rightarrow A(\delta)) \rightarrow A(\theta)$$

for any constant θ, where $\prec$ is the polynomial time computable representation of α.

Remark 2.8. Firstly, note that the theory $\mathrm{TI}(\forall_1, \prec)$ proves

$$\forall \delta \prec \theta (\forall \gamma \prec \delta \; A(\gamma) \rightarrow A(\delta)) \rightarrow \forall \delta \prec \theta A(\delta)$$

where $\theta \in A$ is a constant ordinal. It is enough to use the induction in the system on $B(\delta) = \delta \preceq \theta \rightarrow \forall \eta \prec \delta A(\eta)$.

Secondly, note that it is possible to present the theory $\mathrm{TI}(\forall_1, \prec)$ in the sequent-style calculus by adding the axioms of PV and the following induction rule for any constant $\theta \in A$ to the usual first order sequent calculus:

$$(Ind_\alpha) \; \frac{\Gamma, \forall \gamma \prec \delta \; A(\gamma) \Rightarrow \Delta, A(\delta)}{\Gamma \Rightarrow \Delta, A(\theta)}$$

Note that by the usual methods, it is easy to prove the free-cut elimination theorem to show that if $\Gamma \cup \Delta \subseteq \forall_1$, then if $\Gamma \Rightarrow \Delta$ is provable in $\mathrm{TI}(\forall_1, \prec)$ then it has a $\mathrm{TI}(\forall_1, \prec)$-proof consisting only of $\forall_1$ formulas.

Thirdly, for some practical reasons, sometimes it is useful to change the induction rule by the rule

$$(Ind'_\alpha) \; \frac{\Gamma, \forall \gamma \prec \delta \; A(\gamma) \Rightarrow \Delta, \forall \gamma \prec \delta + 1 \; A(\gamma)}{\Gamma \Rightarrow \Delta, A(\theta)}$$

In the presence of the other first order rules specifically the $\forall_1$-cut rule, the equivalence of these two induction rules is trivial.

Lemma 2.9. $\mathrm{TI}(\forall_1, \prec)$ *is an* $\mathcal{L}_{\mathrm{PV}}$*-extension of the theory* $\mathrm{PRA} + \bigcup_{\beta \prec \alpha} \mathrm{PRWO}(\prec_\beta)$*, i.e., for any* $A \in \mathcal{L}_{\mathrm{PV}}$*, if* $\mathrm{PRA} + \bigcup_{\beta \prec \alpha} \mathrm{PRWO}(\prec_\beta) \vdash A$ *then* $\mathrm{TI}(\forall_1, \prec) \vdash A$.

Proof. Note that the primitive recursive representation of the ordinal α is equivalent to its polynomial time representation provably in PRA we can use this representation in PRWO($\prec_\beta$). Then define $\exists_1$ as the class of formulas logically equivalent to the negations of $\forall_1$ and also define $I\exists_1$ and $I\forall_1$ as PV plus the usual induction on $\exists_1$ and $\forall_1$ formulas, respectively. Note that the primitive recursive functions are $\exists_1$-definable in the theory $I\exists_1$ more or less in the same way that they are represented in $I\Sigma_1$ via Σ_1 formulas. More precisely, for any function symbol f, we claim that there existences a quantifier-free PV-formula $A_f(\vec{x}, w, y)$ such that $\exists w A_f(\vec{x}, w, y)$ plays the role of the definition of f in $I\exists_1$ and w encodes the computation of f on input $\vec{x}$ with the result y. The proof is by induction on the structure of f. The basic functions and composition cases are easy. For the recursion case if $f(\vec{x}, y)$ is defined via recursive equations $f(\vec{x}, 0) = g(\vec{x})$ and $f(\vec{x}, y+1) = h(\vec{x}, y, f(\vec{x}, y))$, define $A_f(\vec{x}, y, \langle u, v\rangle, z)$ as $A_g(\vec{x}, u_0, v_0) \wedge \forall i \leq l(v) A_h(\vec{x}, i, v_i, u_{i+1}, v_{i+1}) \wedge v_{l(v)} = z$ in which v stands for the sequence that encodes all $f(\vec{x}, i)$ for any $i \leq y$, $l(v)$ means the length of this sequence and u encodes the sequence of computations u_i in which u_0 reads $\vec{x}$ and compute $v_0 = f(\vec{x}, 0)$ and u_{i+1} reads $\vec{x}$, i and $f(\vec{x}, i)$ and compute $f(\vec{x}, i+1)$ via the function h. Note that the predicate $\forall i \leq l(v) A_h(\vec{x}, i, v_i, u_i, v_{i+1})$ is polynomial computable since $l(v) \leq |v|$ where $|v|$ is the binary length of v. Hence there exists a polynomial time function symbol in PV like F such that PV proves that $F(\vec{x}, u, v) = 1$ iff $\forall i \leq l(v) A_h(\vec{x}, i, v_i, u_i, v_{i+1})$. Therefore, A_f can be written in a quantifier-free form. The fact that these definitions are functional and total in $I\exists_1$ is similar to what we had for the similar representation in $I\Sigma_1$.

Now note that with the usual technique to show $I\Pi_1 = I\Sigma_1$ we can similarly show that $I\forall_1 = I\exists_1$. Hence all primitive recursive functions are interpretable in $I\forall_1$. Therefore, we can extend the language of TI($\forall_1, \prec$) to the whole language of PRA to extend the transfinite induction even for $\forall_1$ formulas that include PRA-function symbols.

So far, we have handled the equational axioms of PRA. For the induction of PRA, note that the strong induction

$$\forall x(\forall y < x A(y) \to A(x)) \to \forall x A(x)$$

for PRA-quantifier-free formulas A is more stronger than the usual induction in PRA. On the other hand, the function o translates this strong induction in PRA to the transfinite induction up to ω which is available in $TI(\forall_1, \prec)$. Now, as the last step it is enough to prove that TI($\forall_1, \prec$) $\vdash$ PRWO($\prec_\beta$).

Assume

$$\forall y[f(\vec{x}, y+1) \prec f(\vec{x}, y) \wedge f(\vec{x}, y) \in A \wedge f(\vec{x}, y) \prec \beta]$$

Now use $\prec_\beta$-induction to prove $A(\gamma, \vec{x}) = \forall y(f(\vec{x}, y) \neq \gamma)$. For $\gamma = 0$ the claim is clear because if $f(\vec{x}, y) = 0$ then $f(\vec{x}, y+1) \prec 0$ which is impossible. Now if $\forall \delta \prec \gamma A(\delta, \vec{x})$ then if $f(\vec{x}, y) = \gamma$ we have $f(\vec{x}, y+1) \prec \gamma$. But non of the elements below γ is in the image of f, hence we have a contradiction. Therefore, we have $A(\gamma, \vec{x})$. Hence, by transfinite induction on $\prec_\beta$, $\forall \gamma \prec \beta A(\gamma, \vec{x})$ which for $\gamma = f(\vec{x}, 0) \prec \beta$ implies $\forall y(f(\vec{x}, y) \neq f(\vec{x}, 0))$ which is a contradiction. □

Putting the two steps together, we have the following characterization of $\Pi_2^0(\mathcal{L}_{\mathrm{PV}}) = \mathcal{L}_{\mathrm{PV}} \cap \Pi_2^0$ consequences of the theory T:

Corollary 2.10. *Let α be the Π_2^0-ordinal of the theory T with the polynomial time representation $\prec$, then we have $T \equiv_{\Pi_2^0(\mathcal{L}_{\mathrm{PV}})} \mathrm{TI}(\forall_1, \prec)$ i.e., for any $A \in \Pi_2^0(\mathcal{L}_{\mathrm{PV}})$, $T \vdash A$ iff $\mathrm{TI}(\forall_1, \prec) \vdash A$.*

Proof. Note that $\mathrm{TI}(\forall_1, \prec)$ is a sub-theory of $\mathrm{PA} + \bigcup_{\beta \prec \alpha} \mathrm{TI}(\prec_\beta)$ which is Π_2^0-equivalent to T by Theorem 2.4. For the converse use Theorem 2.4 and Theorem 2.9. □

3 Ordinal Flows

In this section we use the notion of an ordinal flow to witness the provable sequents of the theory $\mathrm{TI}(\forall_1, \prec)$. This witnessing leads to the algorithm that we explained in the Introduction.

Definition 3.1. Let $A(\vec{x})$, $B(\vec{x})$ and $H(\delta, \vec{x})$ be some formulas in $\forall_1$. A pair (H, β) where $\beta \prec \alpha$ is called an α-flow if:

(i) $\mathrm{PV} \vdash A(\vec{x}) \leftrightarrow H(0, \vec{x})$.

(ii) $\mathrm{PV} \vdash \forall\, 1 \preceq \delta \prec \beta\, [\forall \gamma \prec \delta\, H(\gamma, \vec{x}) \rightarrow \forall \gamma \prec \delta + 1\, H(\gamma, \vec{x})]$.

(iii) $\mathrm{PV} \vdash H(\beta, \vec{x}) \leftrightarrow B(\vec{x})$.

We denote the existence of an α-flow from A to B by $A \rhd_\alpha B$ and we abbreviate $\bigwedge \Gamma \rhd_\alpha \bigvee \Delta$ by $\Gamma \rhd_\alpha \Delta$. Moreover, when it is clear from the context, we omit the subscript α everywhere.

In order to use the ordinal flows, it is more convenient to develop a high level calculus for this new notion. The following lemmas are devoted to this current task.

Lemma 3.2. *(Conjunction Application) Let $C(\vec{x}) \in \forall_1$ be a formula. If $A(\vec{x}) \rhd B(\vec{x})$ then $A(\vec{x}) \wedge C(\vec{x}) \rhd B(\vec{x}) \wedge C(\vec{x})$.*

Proof. Since $A(\vec{x}) \rhd B(\vec{x})$, then by Definition 3.1 there exist an ordinal β and a formula $H(\gamma, \vec{x}) \in \forall_1$ such that we have the conditions in the Definition 3.1. Define $\beta' = \beta$ and $H'(\gamma, \vec{x}) = H(\gamma, \vec{x}) \wedge C(\vec{x})$. It is clear that the (H', β') is an α-flow from $A(\vec{x}) \wedge C(\vec{x})$ to $B(\vec{x}) \wedge C(\vec{x})$. The main point here is that the fact

$$\mathrm{PV} \vdash \forall\, 1 \preceq \delta \prec \beta\, [\forall \gamma \prec \delta\, H(\gamma, \vec{x}) \to \forall \gamma \prec \delta + 1\, H(\gamma, \vec{x})]$$

implies

$$\mathrm{PV} \vdash \forall\, 1 \preceq \delta \prec \beta\, [\forall \gamma \prec \delta\, H(\gamma, \vec{x}) \wedge C(\vec{x}) \to \forall \gamma \prec \delta + 1\, H(\gamma, \vec{x}) \wedge C(\vec{x})]$$

□

Lemma 3.3. *(Disjunction Application) Let $C(\vec{x}) \in \forall_1$ be a formula. If $A(\vec{x}) \rhd B(\vec{x})$ then $A(\vec{x}) \vee C(\vec{x}) \rhd B(\vec{x}) \vee C(\vec{x})$.*

Proof. The proof is similar to the proof of the Lemma 3.2. □

Lemma 3.4. *(i) (Weak Gluing) If $A(\vec{x}) \rhd B(\vec{x})$ and $B(\vec{x}) \rhd C(\vec{x})$, then $A(\vec{x}) \rhd C(\vec{x})$.*

(ii) (Strong Gluing) If for $\delta \succeq \lambda$ we have $\forall \gamma \prec \delta\, A(\gamma, \vec{x}) \rhd \forall \gamma \prec \delta + 1\, A(\gamma, \vec{x})$, then for any $\theta \succeq \lambda$ we have $\forall \gamma \prec \lambda\, A(\gamma, \vec{x}) \rhd A(\theta, \vec{x})$.

Proof. For (i), since $A(\vec{x}) \rhd B(\vec{x})$ there exist an ordinal β and a formula $H(\gamma, \vec{x}) \in \forall_1$ such that PV proves the conditions in the Definition 3.1. On the other hand since $B(\vec{x}) \rhd C(\vec{x})$ we have the corresponding data for $B(\vec{x})$ to $C(\vec{x})$ which we denote by β' and $H'(\gamma, \vec{x})$. Define $\beta'' = \beta + \beta'$ and

$$H''(\gamma, \vec{x}) = \begin{cases} H(\gamma, \vec{x}) & \gamma \preceq \beta \\ H'(\gamma \dot{-} \beta, \vec{x}) & \beta \prec \gamma \preceq \beta + \beta' \end{cases}$$

It is easy to check that (β'', H'') is an α-flow from $A(\vec{x})$ to $C(\vec{x})$. The reason is simple. First note that $H''(\beta + \beta', \vec{x})$ is equivalent to $H'(\beta', \vec{x})$ which is equivalent to $C(\vec{x})$. Moreover, we have

$$\mathrm{PV} \vdash \forall\, 1 \preceq \delta \prec \beta\, [\forall \gamma \prec \delta\, H''(\gamma, \vec{x}) \to \forall \gamma \prec \delta + 1\, H''(\gamma, \vec{x})]$$

because if $\delta \preceq \beta$ then the claim reduces to the same claim for H. If $\beta \prec \delta$, then it is enough to prove $H''(\delta, \vec{x})$ or equivalently $H'(\delta \dot{-} \beta, \vec{x})$ from

$\forall\gamma \prec \delta\ H''(\gamma, \vec{x})$ which is stronger than $\forall\beta \preceq \gamma \prec \delta\ H''(\gamma, \vec{x})$ or equivalently $\forall\gamma \prec \delta\ H'(\gamma, \vec{x})$.

For (ii) first let us prove $\forall\gamma \prec \lambda\ A(\gamma, \vec{x}) \rhd \forall\gamma \prec \theta + 1\ A(\gamma, \vec{x})$. If we have $\forall\gamma \prec \delta A(\gamma, \vec{x}) \rhd \forall\gamma \prec \delta{+}1\ A(\gamma, \vec{x})$ then there exists β and $H(\eta, \delta, \vec{x})$ such that we have the conditions of the Definition 3.1. Define $\beta' = \beta \times (\theta + 1 \dot{-} \lambda)$ and $I(\tau, \vec{x}) = H(\tau \dot{-} \beta d(\tau, \beta), \lambda + d(\tau, \beta), \vec{x})$. It is easy to see that (I, β') is an α-flow from $\forall\gamma \prec \lambda\ A(\gamma, \vec{x})$ to $\forall\gamma \prec \theta{+}1\ A(\gamma, \vec{x})$. Now since $\forall\gamma \prec \theta{+}1\ A(\gamma, \vec{x})$ implies $A(\theta, \vec{x})$ by weak gluing we have an α-flow from $\forall\gamma \prec \lambda\ A(\gamma, \vec{x})$ to $A(\theta, \vec{x})$ which completes the proof. □

Remark 3.5. Note that if for some formulas $A, B \in \forall_1$ we have $\mathrm{PV} \vdash A \to B$, then we will have $A \rhd B$. It is enough to define $\beta = 1$ and $H(\gamma, \vec{x}) = (\gamma = 0 \to A(\vec{x})) \wedge (\gamma = 1 \to B(\vec{x}))$. Having this observation, by the assumptions $(\mathrm{PV} \vdash A \to B)$, $(B \rhd C)$ and $(\mathrm{PV} \vdash C \to D)$ and by the weak gluing we will have $A \rhd D$. We will use this special case of weak gluing, frequently.

Lemma 3.6. *(Conjunction and Disjunction Rules)*

- (i) *If* $\Gamma, A \rhd \Delta$ *or* $\Gamma, B \rhd \Delta$, *then* $\Gamma, A \wedge B \rhd \Delta$.
- (ii) *If* $\Gamma_0 \rhd \Delta_0, A$ *and* $\Gamma_1 \rhd \Delta_1, B$, *then* $\Gamma_0, \Gamma_1 \rhd \Delta_0, \Delta_1, A \wedge B$.
- (iii) *If* $\Gamma \rhd \Delta, A$ *or* $\Gamma \rhd \Delta, B$, *then* $\Gamma \rhd \Delta, A \vee B$.
- (iv) *If* $\Gamma_0, A \rhd \Delta_0$ *and* $\Gamma_1, B \rhd \Delta_1$, *then* $\Gamma_0, \Gamma_1, A \vee B \rhd \Delta_0, \Delta_1$.

Proof. For (i) and (iii), note that we have $A \wedge B \to A$, $A \wedge B \to B$, $A \to A \vee B$ and $B \to A \vee B$ provable in PV. Then by the Remark 3.5, we have what we wanted.

For (ii) and (iv), we just prove (ii). The proof for (iv) is just dual to the one for (ii). If $\Gamma_0 \rhd \Delta_0, A$, then by definition we have $\bigwedge \Gamma_0 \rhd \bigvee \Delta_0 \vee A$. By conjunction application, we have $\bigwedge \Gamma_0 \wedge \bigwedge \Gamma_1 \rhd (\bigvee \Delta_0 \vee A) \wedge \bigwedge \Gamma_1$. Moreover, we have $\bigwedge \Gamma_1 \rhd \bigvee \Delta_1 \vee B$ and by conjunction application we have

$$\bigwedge \Gamma_1 \wedge (\bigvee \Delta_0 \vee A) \rhd (\bigvee \Delta_1 \vee B) \wedge (\bigvee \Delta_0 \vee A).$$

Therefore, by weak gluing

$$\bigwedge \Gamma_0 \wedge \bigwedge \Gamma_1 \rhd (\bigvee \Delta_1 \vee B) \wedge (\bigvee \Delta_0 \vee A).$$

But we also have

$$\mathrm{PV} \vdash (\bigvee \Delta_1 \vee B) \wedge (\bigvee \Delta_0 \vee A) \to \bigvee \Delta_1 \vee \bigvee \Delta_0 \vee (A \wedge B).$$

Hence by the Remark 3.5 we have

$$\bigwedge \Gamma_0 \wedge \bigwedge \Gamma_1 \rhd \bigvee \Delta_0 \vee \bigvee \Delta_1 \vee (A \wedge B).$$

which means

$$\Gamma_0, \Gamma_1 \rhd \Delta_0, \Delta_1, (A \wedge B).$$

□

Lemma 3.7. *(Cut and Induction Rule)*

(*i*) *If* $\Gamma_0 \rhd \Delta_0, A$ *and* $\Gamma_1, A \rhd \Delta_1$ *then* $\Gamma_0, \Gamma_1 \rhd \Delta_0, \Delta_1$.

(*ii*) *If* $\Gamma, \forall \gamma \prec \delta\ A(\gamma, \vec{x}) \rhd \forall \gamma \prec \delta + 1\ A(\gamma, \vec{x}), \Delta,$ *then* $\Gamma \rhd A(\theta, \vec{x}), \Delta$.

Proof. For (i), Since $\Gamma_0 \rhd \Delta_0, A$ and $\Gamma_1, A \rhd \Delta_1$ then $\bigwedge \Gamma_0 \rhd \bigvee \Delta_0 \vee A$ and $\bigwedge \Gamma_1 \wedge A \rhd \bigvee \Delta_1$. Apply conjunction with $\bigwedge \Gamma_1$ on the first one and disjunction with $\bigvee \Delta_0$ on the second one to prove $\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_0 \rhd (\bigvee \Delta_0 \vee A) \wedge \bigwedge \Gamma_1$ and $(\bigwedge \Gamma_1 \wedge A) \vee \bigvee \Delta_0 \rhd \bigvee \Delta_1 \vee \bigvee \Delta_0$. Since $(\bigvee \Delta_0 \vee A) \wedge \bigwedge \Gamma_1 \rhd (\bigwedge \Gamma_1 \wedge A) \vee \bigvee \Delta_0$, by using gluing we will have $\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_0 \rhd \bigvee \Delta_0 \vee \bigvee \Delta_1$.

For (ii) we reduce the induction case to the strong gluing case. Since

$$\Gamma, \forall \gamma \prec \delta\ A(\gamma, \vec{x}) \rhd \forall \gamma \prec \delta + 1\ A(\gamma, \vec{x}), \Delta$$

define $B(\delta, \vec{x}) = \forall \gamma \prec \delta\ A(\gamma, \vec{x})$. By definition, $\bigwedge \Gamma \wedge B(\delta, \vec{x}) \rhd \bigvee \Delta \vee B(\delta + 1, \vec{x})$. Therefore, by disjunction application we have

$$(\bigwedge \Gamma \wedge B(\delta, \vec{x})) \vee \bigvee \Delta \rhd \bigvee \Delta \vee B(\delta + 1, \vec{x}) \vee \bigvee \Delta$$

and we know

$$\mathrm{PV} \vdash \bigvee \Delta \vee B(\delta + 1, \vec{x}) \vee \bigvee \Delta \to \bigvee \Delta \vee B(\delta + 1, \vec{x}).$$

Hence by the Remark 3.5,

$$(\bigwedge \Gamma \wedge B(\delta, \vec{x})) \vee \bigvee \Delta \rhd \bigvee \Delta \vee B(\delta + 1, \vec{x}).$$

Then by conjunction introduction and the fact that $(\bigwedge \Gamma \wedge B(\delta, \vec{x})) \vee \bigvee \Delta) \rhd \bigwedge \Gamma \vee \bigvee \Delta$,

$$((\bigwedge \Gamma \wedge B(\delta, \vec{x})) \vee \bigvee \Delta), (\bigwedge \Gamma \wedge B(\delta, \vec{x})) \vee \bigvee \Delta) \rhd (\bigvee \Delta \vee B(\delta + 1, \vec{x})) \wedge (\bigwedge \Gamma \vee \bigvee \Delta)$$

Moreover we have

$$\mathrm{PV} \vdash (\phi \vee \psi) \wedge (\sigma \vee \psi) \to (\phi \wedge \sigma) \vee \psi.$$

Hence, by using the contraction which leads to something PV-equivalent with the left side, we have

$$(\bigwedge \Gamma \wedge B(\delta, \vec{x})) \vee \bigvee \Delta \rhd (\bigwedge \Gamma \wedge B(\delta + 1, \vec{x})) \vee \bigvee \Delta.$$

But since for $\delta \neq 0$, the left and the right sides are PV-equivalent to

$$\forall \gamma \prec \delta[(\bigwedge \Gamma \wedge A(\gamma, \vec{x})) \vee \bigvee \Delta]$$

and

$$\forall \gamma \prec \delta + 1[(\bigwedge \Gamma \wedge A(\gamma, \vec{x})) \vee \bigvee \Delta]$$

respectively, for $\delta \neq 0$ we have

$$\forall \gamma \prec \delta[(\bigwedge \Gamma \wedge A(\gamma, \vec{x})) \vee \bigvee \Delta] \rhd \forall \gamma \prec \delta + 1[(\bigwedge \Gamma \wedge A(\gamma, \vec{x})) \vee \bigvee \Delta]$$

hence by strong gluing we will have

$$\forall \gamma \prec 1(\bigwedge \Gamma \wedge A(1, \vec{x})) \vee \bigvee \Delta] \rhd (\bigwedge \Gamma \wedge A(\theta, \vec{x})) \vee \bigvee \Delta]$$

Since the left side is PV-equivalent to $(\bigwedge \Gamma \wedge A(0, \vec{x})) \vee \bigvee \Delta]$ we have

$$(\bigwedge \Gamma \wedge A(0, \vec{x})) \vee \bigvee \Delta] \rhd (\bigwedge \Gamma \wedge A(\theta, \vec{x})) \vee \bigvee \Delta]$$

By assumption we have $\Gamma \rhd A(0, \vec{x}), \Delta$ and since $\Gamma \rhd \bigwedge \Gamma$ by propositional rules

$$\Gamma, \Gamma \rhd (\bigwedge \Gamma \wedge A(0, \vec{x})), \Delta$$

Since Γ, Γ is PV-equivalent to Γ, we have

$$\Gamma \rhd (\bigwedge \Gamma \wedge A(0, \vec{x})) \vee \bigvee \Delta$$

and by weak gluing

$$\Gamma \rhd (\bigwedge \Gamma \wedge A(\theta, \vec{x})) \vee \bigvee \Delta]$$

and since the right side implies $A(\theta, \vec{x})) \vee \bigvee \Delta$ in PV we have $\Gamma \rhd A(\theta, \vec{x}), \Delta$. □

Lemma 3.8. *(Negation and Implication Rules)*

(i) *If* $\Gamma \rhd \Delta, A$ *then* $\Gamma, \neg A \rhd \Delta$.

(ii) *If* $\Gamma, A \rhd \Delta$ *then* $\Gamma \rhd \Delta, \neg A$.

(iii) *If* $\Gamma_0 \rhd \Delta_0, A$ *and* $\Gamma_1, B \rhd \Delta_1$ *then* $\Gamma_0, \Gamma_1, A \to B \rhd \Delta_0, \Delta_1$.

(iv) *If* $\Gamma, A \rhd \Delta, B$ *then* $\Gamma \rhd \Delta, A \to B$.

Proof. For (i), since $\Gamma \rhd \Delta, A$ by conjunction application $\bigwedge \Gamma \wedge \neg A \rhd (\bigvee \Delta \vee A) \wedge \neg A$. Since $(\bigvee \Delta \vee A) \wedge \neg A$ implies $\bigvee \Delta$ we have

$$(\bigvee \Delta \vee A) \wedge \neg A \rhd \bigvee \Delta$$

and hence by weak gluing $\Gamma, \neg A \rhd \Delta$. The proof for (ii) is similar. For (iii) and (iv), note that $A \to B$ and $\neg A \vee B$ are equivalent in PV and hence we have $A \to B \rhd \neg A \vee B$ and $\neg A \vee B \rhd A \to B$. Then by cut it is possible to reduce (iii) and (iv) to the same things for $\neg A \vee B$. But these claims are provable by negation and disjunction rules. □

Theorem 3.9. *(Soundness) If* $\Gamma \cup \Delta \subseteq \forall_1$ *and* $\mathrm{TI}(\forall_1, \prec) \vdash \Gamma \Rightarrow \Delta$, *then there exists an* α*-flow from* Γ *to* Δ.

Proof. We prove the lemma by induction on the length of the proof of $\Gamma(\vec{x}) \Rightarrow \Delta(\vec{x})$ using the induction rule mentioned in the Remark 2.8. Note that the proof consists only of $\forall_1$ formulas by definition.

1. (Axioms). If $\Gamma(\vec{x}) \Rightarrow \Delta(\vec{x})$ is a logical axiom then the claim is trivial. If it is a non-logical axiom then the claim will be also trivial because all non-logical axioms are provable in PV. Therefore there is nothing to prove.

2. (Structural Rules). These are derivable from the same rules available in PV.

3. (Cut). See the Lemma 3.7.

4. (Propositional). The conjunction and disjunction cases are proved in the Lemma 3.6. The implication and negation cases are proved in the Lemma 3.8.

5. (Universal Quantifier, Right). If $\Gamma(\vec{x}) \Rightarrow \Delta(\vec{x}), \forall z B(\vec{x}, z)$ is proved by the $\forall R$ rule by $\Gamma(\vec{x}) \Rightarrow \Delta(\vec{x}), B(\vec{x}, z)$, then by IH, $\Gamma(\vec{x}) \rhd \Delta(\vec{x}), B(\vec{x}, z)$. Therefore, there exist an ordinal β and a formula $H(\gamma, \vec{x}, z) \in \forall_1$ such that the conditions of the Definition 3.1 are provable in PV. Define $\beta' = \beta$ and $H'(\gamma, \vec{x}) = \forall z H(\gamma, \vec{x}, z)$. Since $H(\gamma, \vec{x}, z) \in \forall_1$ we have $\forall z H(\gamma, \vec{x}, z) \in \forall_1$. The other conditions to ensure that the new sequence is an α-flow from

$\forall z[\bigwedge \Gamma(\vec{x})]$ to $\forall z[B(\vec{x}, z) \vee \bigvee \Delta]$ is a straightforward consequence of the fact that if

$$\mathrm{PV} \vdash \forall \gamma \prec \delta H(\gamma, z, \vec{x}) \to \forall \gamma \prec \delta + 1 \; H(\gamma, z, \vec{x}),$$

then

$$\mathrm{PV} \vdash \forall \gamma \prec \delta \forall z H(\gamma, z, \vec{x}) \to \forall \gamma \prec \delta + 1 \forall z H(\gamma, z, \vec{x}).$$

Finally, note that $\Gamma \cup \Delta$ does not have a free z variable and hence $\forall z[\bigwedge \Gamma]$ and $\forall z[B(\vec{x}, z) \vee \bigvee \Delta]$ are equivalent to $\bigwedge \Gamma$ and $\bigvee \Delta \vee \forall z B(\vec{x}, z)$, provably in PV which completes the proof.

6. (Universal Quantifier, Left). If $\Gamma(\vec{x}), \forall z B(\vec{x}, z) \Rightarrow \Delta(\vec{x})$ is proved by the $\forall L$ rule by $\Gamma(\vec{x}), B(\vec{x}, s(\vec{x})) \Rightarrow \Delta(\vec{x})$, then since $\mathrm{PV} \vdash \forall z B(\vec{x}, z) \to B(\vec{x}, s(\vec{x}))$, and

$$\Gamma(\vec{x}), B(\vec{x}, s(\vec{x})) \rhd \Delta(\vec{x}),$$

we have

$$\Gamma(\vec{x}), \forall z B(\vec{x}, z) \rhd \Delta(\vec{x}).$$

7. (Induction). See the Lemma 3.7.

□

And also like in the bounded case we have the completeness theorem:

Theorem 3.10. *(Completeness) If* $\Gamma \cup \Delta \subseteq \forall_1$ *and* $\Gamma \rhd \Delta$, *then* $\mathrm{TI}(\forall_1, \prec) \vdash \Gamma \Rightarrow \Delta$.

Proof. If there exists an α-flow from Γ to Δ then it means that there exists (H, β) such that

(*i*) $\mathrm{PV} \vdash \bigwedge \Gamma(\vec{x}) \leftrightarrow H(0, \vec{x})$.

(*ii*) $\mathrm{PV} \vdash \forall \, 1 \preceq \delta \prec \beta \; [\forall \gamma \prec \delta \; H(\gamma, \vec{x}) \to \forall \gamma \prec \delta + 1 \; H(\gamma, \vec{x})]$.

(*iii*) $\mathrm{PV} \vdash H(\beta, \vec{x}) \leftrightarrow \bigvee \Delta(\vec{x})$.

Therefore, using induction on $H(\delta, \vec{x})$ we have

$$\mathrm{TI}(\forall_1, \prec) \vdash H(0, \vec{x}) \Rightarrow H(\beta, \vec{x}),$$

and thus $\mathrm{TI}(\forall_1, \prec) \vdash \bigwedge \Gamma(\vec{x}) \Rightarrow \bigvee \Delta(\vec{x})$. □

Therefore, we have the following corollary as the characterization of the implications of the low complexity universal statements based on an ordinal-length sequence of PV implications that can be also witnessed by the polynomial time computable functions in PV.

Definition 3.11. Let $A(\vec{x}, y)$ be a polynomial time computable predicate, α be an ordinal with a polynomial time representation and $\beta \prec \alpha$. Then by a PLS($\prec_\beta$) program for A we mean the following data:

(*i*) An initial sequence of polynomial time functions $\vec{i}(\vec{x})$,

(*ii*) A polynomial time predicate $G(\gamma, \vec{x}, \vec{z})$ which intuitively means that $\vec{z}$ is a feasible solution for the input $\vec{x}$,

(*iii*) A sequence of polynomial time functions $\vec{N}(\gamma, \vec{x}, \vec{z})$,

(*iv*) A sequence of polynomial time functions $q(\gamma, \vec{x}, \vec{z})$,

(*v*) A polynomial time computable function $p(\vec{x}, \vec{z})$.

such that:

(*i*) $\mathrm{PV} \vdash G(\beta, \vec{x}, \vec{i}(\vec{x}))$,

(*ii*) $\mathrm{PV} \vdash \gamma \neq 0 \to q(\gamma, \vec{x}, \vec{z}) \prec \gamma$,

(*iii*) $\mathrm{PV} \vdash \gamma \neq 0 \to [G(\gamma, \vec{x}, \vec{z}) \to G(q(\gamma, \vec{x}, \vec{z}), \vec{x}, \vec{N}(\gamma, \vec{x}, \vec{z}))]$,

(*iv*) $\mathrm{PV} \vdash G(0, \vec{x}, \vec{z}) \to A(\vec{x}, p(\vec{x}, \vec{z}))$.

By PLS($\preceq_\beta$) we mean the class of all formulas $\forall \vec{x} \exists y A(\vec{x}, y)$ such that there exists a PLS($\preceq_\beta$) program for A.

Theorem 3.12. *Let $A(\vec{x}, y)$ be a polynomial time computable predicate and T an arithmetical theory with the proof theoretic ordinal α_T with a polynomial time representation. Then $T \vdash \forall \vec{x} \exists y A(\vec{x}, y)$ iff there exists $\beta \prec \alpha_T$ and a* PLS($\preceq_\beta$) *program for A.*

Proof. First it is clear that the existence of a PLS($\preceq_\beta$) program for A implies the existence of an α-flow from $\forall y \neg A(\vec{x}, y)$ to $\bot$ and hence $T \vdash \forall y \neg A(\vec{x}, y) \to \bot$ which prove the claim. For the converse, assume that $T \vdash \forall \vec{x} \exists y A(\vec{x}, y)$ where $A(\vec{x}, y)$ is quantifier-free in the language of PV. Then we have $\forall y \neg A(\vec{x}, y) \rhd \bot$. Hence there exists (H, β) such that:

(*i*) $\mathrm{PV} \vdash \forall y \neg A(\vec{x}, y) \to H(0, \vec{x})$.

(*ii*) $\mathrm{PV} \vdash \forall\, 1 \preceq \gamma \prec \beta\, [\forall \delta \prec \gamma\; H(\delta, \vec{x}) \to \forall \delta \prec \gamma + 1\; H(\delta, \vec{x})]$.

(*iii*) $\mathrm{PV} \vdash H(\beta, \vec{x}) \to \bot$.

Since $H \in \forall_1$ we have $H(\gamma, \vec{x}) \equiv_{\mathrm{PV}} \forall \vec{z} G(\gamma, \vec{x}, \vec{z})$ where G is quantifier-free. On the other hand, all the conditions are provable in PV which means that we can witness the existential quantifiers by polynomial time functions. Hence, there are polynomial time functions $Y(\vec{x}, \vec{z})$, $\vec{Z}(\vec{x}, \vec{z}, \delta)$, $\Delta(\vec{x}, \vec{z}, \delta)$ and $\vec{W}(\vec{x})$ such that:

(i') $\mathrm{PV} \vdash \neg A(\vec{x}, Y(\vec{x}, \vec{z})) \to G(0, \vec{x}, \vec{z})$.

(ii') $\mathrm{PV} \vdash \forall\, 1 \preceq \gamma \prec \beta\, [\Delta(\vec{x}, \vec{z}, \delta) \prec \gamma \to G(\Delta(\vec{x}, \vec{z}, \delta), \vec{x}, \vec{Z}(\vec{x}, \vec{z}, \delta)) \to \delta \prec \gamma + 1 \to G(\delta, \vec{x}, \vec{z})]$.

(iii') $\mathrm{PV} \vdash G(\beta, \vec{x}, \vec{W}(\vec{x})) \to \bot$.

Put $\delta = \gamma$ in (ii'), then we have

$$\mathrm{PV} \vdash \forall \gamma \prec \beta\, [(\Delta(\vec{x}, \vec{z}, \gamma) \prec \gamma \to G(\Delta(\vec{x}, \vec{z}, \gamma), \vec{x}, \vec{Z}(\vec{x}, \vec{z}, \gamma)) \to G(\gamma, \vec{x}, \vec{z})].$$

Define

$$q(\vec{x}, \gamma, \vec{z}) = \begin{cases} \Delta(\vec{x}, \vec{z}, \gamma) & \text{if } \neg G(\vec{x}, \gamma, \vec{z}) \\ 0 & \text{if } G(\vec{x}, \gamma, \vec{z}) \end{cases}$$

and $\vec{i}(\vec{x}) = \vec{W}(\vec{x})$. It is easy to see that this new data is a $\mathrm{PLS}(\preceq_\beta)$ program for A. □

We can use the ordinal PLS programs to characterize the total NP search problems of any theory with the proof theoretic ordinal α.

Definition 3.13. By a total NP-search problem of a theory T, we mean all consequences of T of the form $\forall \vec{x} \exists y A(\vec{x}, y)$ where A is a polynomial time computable predicate and $\mathrm{PV} \vdash A(\vec{x}, y) \to |y| \leq p(|\vec{x}|)$ where p is a polynomial. We denote the class of all these formulas by $\mathrm{TFNP}(T)$ and the class $\mathrm{PLS}(\preceq_\beta) \cap \mathrm{TFNP}(\mathbb{N})$ by $\mathrm{PLS}^b(\preceq_\beta)$.

We believe that the notation system introduced in [2] actually provides a polynomial time representation of the ordinal ϵ_0. Given this fact, as a corollary we will have:

Corollary 3.14. *(i) ([1])* $\mathrm{TFNP}(\mathrm{PA}) = \bigcup_{\beta \prec \epsilon_0} \mathrm{PLS}^b(\preceq_\beta)$.

(ii) Let α be an ordinal and $\epsilon(\alpha)$ be the least ϵ number after α with a polynomial-time representation. Then

$$\mathrm{TFNP}(\mathrm{PA} + \mathrm{TI}(\alpha)) = \bigcup_{\beta \prec \epsilon(\alpha)} \mathrm{PLS}^b(\preceq_\beta)$$

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