Mathematical Structuralism: Exercises

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Exercise 1. Show that the identity map of a given object is unique, i.e., a map $f : A \to A$ such that $fg = g$ and $hf = h$, for any $g : B \to A$ and $h : A \to C$.

Exercise 2. Prove that the inverse of a map is unique. Hence, it is well-defined to denote the inverse of f by f^{-1}.

Exercise 3. Prove that $id_A : A \to A$ is an isomorphism and if $f : A \to B$ and $g : B \to C$ are isomorphisms, then so is $g \circ f : A \to C$.

Exercise 4. Prove that in **Set**, the isomorphisms are the bijective maps. Show that in the category **Poset**, there are morphisms that are also bijections, but not isomorphisms. What are the isomorphisms in posets, monoids, $\mathbf{Set}^{\mathcal{C}}$?

Exercise 5. Show that if $f : A \to B$ is an isomorphism in $\mathcal{C}$, then it is also an isomorphism in $\mathcal{C}^{op}$. Use this fact to show that the dual of a groupoid is also a groupoid.

Exercise 6. Let $\mathcal{C}$ be a category. Show that the collection of isomorphisms in $\mathcal{C}$ defines a subcategory, the maximal groupoid inside $\mathcal{C}$.

A morphism $f : A \to B$ is called monic if $fg = fh$ implies $g = h$, for any $g, h : C \to A$. Dually a morphism $f : A \to B$ is called epic if $gf = hf$ implies $g = h$, for any $g, h : B \to C$.

Exercise 7. Show that in a poset all maps are monic and epic.

Exercise 8. Show that the monic (epic) maps in **Set**, **Top** and **Ab** are the injective (surjective) maps.

Exercise 9. Show that the monic maps in **Mon** and **Ring** are the injective maps, but not all epics are surjective. (Hint: Think about inclusions $\mathbb{N} \to \mathbb{Z}$ and $\mathbb{Z} \to \mathbb{Q}$, respectively.)

Exercise 10. Show that the epic maps in **Grp** are the surjective homomorphisms.

Exercise 11. Show that the epics in the category of Hausdorff spaces are the continuous maps whose range are dense in their codomain.

Exercise 12. What are the monic maps in **Rel**?

Exercise 13. What are the monics and epics in $\mathbf{Set}^{\mathcal{C}}$, for a small category $\mathcal{C}$?

Exercise 14. What are the monics and epics in the category of fields?

Exercise 15. Show that a functor is a monic in **Cat** iff it is injective both on objects and morphisms. Prove that the corresponding statement for epics in **Cat** does not hold.

Exercise 16. Show that any isomorphism is both monic and epic. Does the converse hold? (Hint: Think of posets. For a more interesting example, note that the inclusion $\mathbb{Z} \to \mathbb{Q}$ in **Ring** is both monic and epic, but it is not isomorphism).

Exercise 17. Show that if fg is monic, then g is also monic. If fg is epic, then f is also epic.

Exercise 18. A morphism $f : A \to A$ is called split if there are $g : A \to B$ and $h : B \to A$ such that $hg = f$ and $gh = id_B$. Show that any split morphism is idempotent, i.e., $ff = f$ and an idempotent map f is split if there are a monic h and an epic g such that $f = hg$. Prove that in **Set** all idempotents are split while it is not generally true.

A morphism $f : A \to B$ is called split monic if there exists $g : B \to A$ such that $gf = id_A$. Dually a morphism $f : A \to B$ is called split epic if there exists $g : B \to A$ such that $fg = id_B$.

Exercise 19. What are the split monics and the split epics in **Set**? Are all epics in $\mathbf{Set}^{\to}$ split epic?

Exercise 20. Show that all monics in $\mathbf{Vec}_{\mathbb{R}}$ are split monic.

Exercise 21. Show that a map $r : G \to H$ is split monic in **Ab** iff there exist abelian groups J and K and isomorphisms $i : J \to G$ and $j : J \oplus K \to H$ such that:

$$\begin{tikzcd} J \arrow[r, "a \mapsto (a,0)"] \arrow[d, "i"'] & J \oplus K \arrow[d, "j"] \\ G \arrow[r, "r"'] & H \end{tikzcd}$$

Similarly, show that a map $s : G \to H$ is split epic in **Ab** iff there exist abelian groups J and K and isomorphisms $i : J \oplus K \to G$ and $j : J \to H$ such that:

$$\begin{tikzcd} J \oplus K \arrow[r, "(a,b)\mapsto a"] \arrow[d, "i"'] & J \arrow[d, "j"] \\ G \arrow[r, "s"'] & H \end{tikzcd}$$

Exercise 22. Show that a morphism $f : A \to B$ is split epic in $\mathcal{C}$ if and only if for all object C of $\mathcal{C}$, the post-composition function $f_* : Hom(C, A) \to Hom(C, B)$ is surjective. Dually show that $f : A \to B$ is split monic if and only if for all objects C in $\mathcal{C}$, the pre-composition function $f^* : Hom(B, C) \to Hom(A, C)$ is surjective.

Exercise 23. Show that any split monic is monic and any split epic is epic. Does the converse hold? (Hint: Think about posets. For a more interesting example, note that the inclusion $\mathbb{Z} \to \mathbb{Q}$ in **Ring** is both monic and epic, but there is no map $\mathbb{Q} \to \mathbb{Z}$).

Exercise 24. Prove that a morphism that is both a monic and a split epic is necessarily an isomorphism. Show that a split monic that is an epic is also an isomorphism.

Exercise 25. Let $\mathcal{F}$ be a collection of objects of a category $\mathcal{C}$. We say that $\mathcal{C}$ has enough $\mathcal{F}$-points when for any $f, g : X \to Y$, if for any object C in $\mathcal{F}$ and any map $h : C \to X$ we have $fh = gh$, then $f = g$. Show that:

- **Set** has enough 1-points.
- **Set**$^{\to}$ does not have enough 1-points.
- **Mon** does not have enough $\{e\}$-points.
- **Grp** does not have enough $\{e\}$-points.
- **Mon** has enough $\mathbb{N}$-points.
- **Grp** has enough $\mathbb{Z}$-points.
- **Quiv** does not have enough 1-points.
- **Quiv** has enough $\{\bullet, (\bullet \to \bullet)\}$-points.
- **Cat** has enough **2**-points.

Exercise 26. Let $\mathcal{F}$ be a collection of objects of a category $\mathcal{C}$. We say that $\mathcal{C}$ has enough $\mathcal{F}$-fibers when for any $f, g : X \to Y$, if any object C in $\mathcal{F}$ and any map $h : Y \to C$ we have $hf = hg$, then $f = g$. Show that:

- **Set** has enough $\{0, 1\}$-fibers.
- **Poset** has enough $\{0 \leq 1\}$-fibers.
- The subcategory of powersets of **Poset** has enough $\{0, 1\}$-fibers.
- $\mathbf{Vec}_{\mathbb{R}}$ has enough $\mathbb{R}$-fibers.

Exercise 27. Any category $\mathcal{C}$ determines a preorder $P(\mathcal{C})$ by defining a binary relation $\leq$ on the objects by $A \leq B$ if and only if there is an arrow $A \to B$. Show that P determines a functor from categories to preorders, by defining its effect on functors between categories and checking the required conditions. Show that P is a (one-sided) inverse to the evident inclusion functor of preorders into categories.

Exercise 28. Show that the construction of the set of conjugacy classes of elements of a group is functorial, defining a functor $Conj : \mathbf{Grp} \to \mathbf{Set}$. Conclude that any pair of groups whose sets of conjugacy classes of elements have differing cardinalities cannot be isomorphic.

Exercise 29. Show that $\mathcal{C}/A$ and $\mathcal{C}/B$ can be isomorphic without A and B being isomorphic.

Exercise 30. Show that $(A/\mathcal{C})^{op} \cong \mathcal{C}^{op}/A$.

Exercise 31. Show that:

- $P(X)/A \cup B \cong P(X)/A \times P(X)/B$ as slices of $(P(X), \subseteq)$.
- $\mathbf{Set}/A + B \cong \mathbf{Set}/A \times \mathbf{Set}/B$.

Exercise 32. Show the followings:

- For any groupid $\mathcal{G}$ we have $\mathcal{G} \cong \mathcal{G}^{op}$.
- $\mathbf{Rel} \cong \mathbf{Rel}^{op}$.
- $(P(X), \subseteq) \cong (P(X), \subseteq)^{op}$.
- It is not necessarily the case that $M \cong M^{op}$, for any monoid M.
- $\mathbf{Set} \ncong \mathbf{Set}^{op}$.

Exercise 33. Show that the image of a functor is not necessarily a subcategory.

Exercise 34. Describe all functors from **Set** to a poset.

Exercise 35. Show that there is no functor $Z : \mathbf{Grp} \to \mathbf{Grp}$ that maps any group to its center. (Hint: Use $S_2 \to S_3 \to S_2$, where S_i's are permutation groups).

Exercise 36. Show that the powerset functor $P : \mathbf{Set} \to \mathbf{Set}$ is faithful but not full.

Exercise 37. The product functor $(-) \times (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ is faithful but not full.

Exercise 38. Show that if $F : \mathcal{C} \to \mathcal{D}$ is full and faithful, then for any two objects A and B in $\mathcal{C}$, we have $A \cong B$ iff $F(A) \cong F(B)$.

Exercise 39. Show that if $F : \mathcal{C} \to \mathcal{D}$ is faithful and $F(f)$ is a monic in $\mathcal{D}$, then f is a monic in $\mathcal{C}$. Show that the same also holds for epics. Conclude that in any concrete category, any morphism that defines an injection of underlying sets is a monic and any morphism that defines a surjection of underlying sets is an epic.

Exercise 40. Find an example to show that a faithful functor need not preserve epics or monics.

Exercise 41. Show that any functor preserves split monic and split epic.

Exercise 42. Prove that $\alpha : ((-) \times (-)) \times (-) \Rightarrow (-) \times ((-) \times (-))$ defined by $\alpha_{A,B,C} : (A \times B) \times C \to A \times (B \times C)$ such that $\alpha_{A,B,C}((a,b),c) = (a,(b,c))$ is a natural isomorphism, where $(-) \times (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$.

Exercise 43. Prove that $\alpha : p_1 \Rightarrow Hom(-,-)$ defined by $\alpha_{A,B} : A \to Hom(B,A)$ such that $\alpha_{A,B}(a) = cons_{A,B,a}$ is a natural transformation, where $p_1 : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$ is the projection on the second element functor and $cons_{A,B,a} : B \to A$ maps every element in B to a.

Exercise 44. Prove that $\alpha : (-)^{(-)+(-)} \Rightarrow (-)^{(-)} \times (-)^{(-)}$ defined by $\alpha_{A,B,C}(g) = (g|_B, g|_C)$ is a natural isomorphism, where $(-) \times (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$, $(-) + (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ and $(-)^{(-)} : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$.

Exercise 45. Prove that $\alpha : ((-) \times (-))^{(-)} \Rightarrow (-)^{(-)} \times (-)^{(-)}$ defined by $\alpha_{A,B,C} : (A \times B)^C \to A^C \times B^C$ such that $\alpha_{A,B,C}(g) = (p_0 \circ g, p_1 \circ g)$ is a natural isomorphism, where $p_0 : A \times B \to A$ and $p_1 : A \times B \to B$ are the projection functions and $(-) \times (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ and $(-)^{(-)} : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$.

Exercise 46. Prove that $\alpha : Hom((-),(-)^{(-)}) \Rightarrow Hom(- \times -, -)$ defined by $\alpha_{A,B,C} : Hom(A, C^B) \to Hom(A \times B, C)$ such that $\alpha_{A,B,C}(g) = \hat{g}$ is a natural isomorphism, where $\hat{g} : A \times B \to C$ maps (a,b) to $g(a)(b)$ and $(-) \times (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ and $(-)^{(-)} : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$.

Exercise 47. Show that the assignments $\alpha_X, \beta_X : P(X) \to P(X)$ with the definition $\alpha_X(A) = X$ and $\beta_X(A) = X - A$ are not natural transformations from $P : \mathbf{Set} \to \mathbf{Set}$ to itself.

Exercise 48. Prove that there is no natural transformation from $id_{(\mathbb{Z},+)}$ to $-id_{(\mathbb{Z},+)}$.

Exercise 49. Show that a natural transformation $\alpha : F \Rightarrow G$ is a natural isomorphism iff α_A is an isomorphism and the inverses of the component morphisms define the components of a natural isomorphism $\alpha^{-1} : G \Rightarrow F$.

Exercise 50. Using direct computation, show that any natural transformation $Hom(-, A) \Rightarrow Hom(-, B)$ is y_f for some $f : A \to B$.

Exercise 51. Let $\mathcal{C}$ be a category. By the center of $\mathcal{C}$, denoted by $Z(\mathcal{C})$, we mean the class of all natural transformation $\alpha : id_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$. Show that for any group G considered as a category, $Z(G)$ corresponds to the set $\{g \in G \mid \forall h \in G \; gh = hg\}$. Use this characterization to show that for any non-trivial abelian groups G and H, if $G \simeq H$ and $F : G \to H$ is an isomorphism, there are at least two different natural transformations over F. Moreover, find a group G such that between any two isomorphisms over G, there is at most one morphism.

Exercise 52. Consider the poset $(\mathbb{Z} + \mathbb{Z}, \leq)$, where $\leq$ is the usual order on each component. Then, take the isomorphism $F = [+1, -1] : (\mathbb{Z} + \mathbb{Z}, \leq) \to (\mathbb{Z} + \mathbb{Z}, \leq)$ define by $F(0, a) = a + 1$ and $F(1, b) = b - 1$. Prove that there is no natural transformations $\alpha : id_{(\mathbb{Z}+\mathbb{Z},\leq)} \Rightarrow F$ and $\beta : F \Rightarrow id_{(\mathbb{Z}+\mathbb{Z},\leq)}$.

Exercise 53. Let $List : \mathbf{Set} \to \mathbf{Set}$ be the functor mapping any set X to the set of all finite sequences of the elements of X and mapping any function $f : X \to Y$ to the function $List(f) : List(X) \to List(Y)$ defined by $List(f)(\sigma_0 \cdots \sigma_n) = f(\sigma_0) \cdots f(\sigma_n)$. Show that the assignment $i : \Delta_1 \to List$ defined by $i_X : \{0\} \to List(X)$ as $i_X(0) = \epsilon$ is a natural transformation, where ϵ is the sequence with the length zero. Moreover, show that the assignment $m : List \times List \to List$ defined by $m_X : List(X) \times List(X) \to List(X)$ as the concatenation operation is a natural transformation.

Exercise 54. Prove that if $\mathcal{E}$ and $\mathcal{F}$ are two categories isomorphic to $\mathbf{Set}$ and $F_0, F_1 : \mathcal{E} \to \mathcal{F}$ be two isomorphisms, then there exists exactly one natural transformation from F_0 to F_1.

Exercise 55. Find all natural transformations from the powerset functor $P : \mathbf{Set} \to \mathbf{Set}$ to itself. Do the same for $P^\circ : \mathbf{Set}^{op} \to \mathbf{Set}$.

Exercise 56. Do there exist any non-identity natural endomorphisms of the category of spaces? That is, does there exist any family of continuous maps $X \to X$, defined for all spaces X and not all of which are identities, that are natural in all maps in the category **Top**?

Exercise 57. Which of the following functors are corepresentable:

- The forgetful functor $F : \mathbf{Mon} \to \mathbf{Set}$
- The forgetful functor $F : \mathbf{Grp} \to \mathbf{Set}$.
- The functor $T_n : \mathbf{Grp} \to \mathbf{Set}$ mapping any group G to $\{x \in G \mid x^n = e\}$.
- The forgetful functor $F : \mathbf{Set}^{\to} \to \mathbf{Set}$, forgetting the source.
- The constant functor $\Delta_1 : \mathbf{Set} \to \mathbf{Set}$.
- The functor $(-)^n : \mathbf{Set} \to \mathbf{Set}$.
- The functor $(-) + 1 : \mathbf{Set} \to \mathbf{Set}$.
- The constant functor $\Delta_2 : \mathbf{Set} \to \mathbf{Set}$.
- The forgetful functor $F : \mathbf{Set}^{\to} \to \mathbf{Set}$, forgetting the target.

Exercise 58. Let $U : \mathbf{Cat} \to \mathbf{Set}$ be the functor that sends a small category to the set of all its maps. Prove that U is corepresentable.

Exercise 59. Prove that if $F : \mathcal{C} \to \mathbf{Set}$ is corepresentable, then F preserves monics, i.e., sends every monic in $\mathcal{C}$ to an injective function. Use the contrapositive to find a covariant set-valued functor defined on your favourite concrete category that is not representable.

Exercise 60. Use the Yoneda lemma to explain the connection between homeomorphisms of the standard unit interval $I = [0, 1]$ and natural automorphisms of the path functor $Path : \mathbf{Top} \to \mathbf{Set}$ called *re-parameterizations*.

Exercise 61. Show that $\mathcal{C}$ is discrete iff for any category $\mathcal{D}$, any map from the objects of $\mathcal{C}$ to the objects of $\mathcal{D}$ gives rise to a unique functor.

Exercise 62. Find a quiver Ω such that $Hom(Q, \Omega) \cong Sub(Q)$, where $Sub(Q)$ is the functor computing subquivers of Q.

Exercise 63. For any quivers P and Q, provide a quiver R such that $Hom(X, R) \cong Hom(X \times P, Q)$.

Exercise 64. Show that any map coming out of a terminal object is monic and any map going into an initial object is epic. Show that any map from a terminal to an initial one is an isomorphism.

Exercise 65. Show that **Grp** and **Rel** have all coproducts.

Exercise 66. Show that the projection (injection) maps in a product (coproduct) is not necessarily epic (monic).

Exercise 67. Show that a functor needs not to preserve terminal or initial objects. The same is true for products or coproducts.

Exercise 68. In **Ab**, the object H^G consisting of all homomorphisms from the group G to H with the pointwise addition is a natural candidate to be the exponential object of H by G. Find out why it does not work.

Exercise 69. Show that the category of small groupoids has all exponentials. Compare the situation with **Grp** and investigate how having more objects is needed to have exponentials.

Exercise 70. Show that in the categories **Rel** and **Mon** the distributivity law, i.e., $A \times (B + C) \cong A \times B + A \times C$ does not hold. The same also holds for the poset of subgroups of $G \times G$, where G is an abelian group. Conclude that these categories do not have all exponentials.

Exercise 71. Show that **Grp** has all coequalizers.

Exercise 72. Show that the equalizer of two functors $F, G : \mathcal{C} \to \mathcal{D}$ is the subcategory of $\mathcal{C}$ consisting of objects and maps over which F and G are equal together with inclusion.

Exercise 73. Show that two maps $f, g : A \to B$ are equal iff their equalizers exists and is an isomorphism. The same also holds for coequalizers.

A map $f : A \to B$ is called regular monic if there are two maps $g, h : B \to C$ such that f is the equlizer of g and h. Dually, a map $f : A \to B$ is called regular epic if there are two maps $g, h : C \to A$ such that f is the coequlizer of g and h.

Exercise 74. What are the regular monics and regular epics in **Set**, **Ab** and **Top**?

Exercise 75. Show that any regular monic (epic) is split monic (epic). Prove that the converse does not hold.

Exercise 76. Show that if all equalizers exist in a category, then the equalizer for any finite family exists, i.e., for any family $\{f_i : A \to B\}_{i=1}^{n}$, there exists a map $e : C \to A$ such that $f_i e = f_j e : C \to B$ for any $1 \leq i, j \leq n$ and for any map $g : D \to A$ such that for any $1 \leq i, j \leq n$ we have $f_i g = f_j g : C \to B$, there exists a unique map $h : D \to C$ such that $eh = g$, i.e.,

$$\begin{tikzcd}
D \arrow[d, "h"'] \arrow[dr, "g"] & & \\
C \arrow[r, "e"'] & A \arrow[r, shift left=2, "f_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "f_n"'] & B
\end{tikzcd}$$

Exercise 77. Show that if the following diagram

$$C \xrightarrow{h} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is equalizer and $i : B \to D$ is monic, then the following diagram

$$C \xrightarrow{h} A \underset{ig}{\overset{if}{\rightrightarrows}} D$$

is also equalizer. Dually, if the following diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C$$

is coequalizer and $i : D \to A$ is epic, then the following diagram

$$D \underset{gi}{\overset{fi}{\rightrightarrows}} B \xrightarrow{h} C$$

is also coequalizer.

Exercise 78. Show that if all equalizers exist, then any idempotent is split, in the sense of Exercise 18. The same is true if all coequalizers exist.

Exercise 79. Show that the pullback of any monic is monic and the pushout of any epic is epic. Show that the former (latter) part of the claim does not hold for pushouts (pullbacks).

Exercise 80. Show that the pullback of a split epic is split epic.

Exercise 81. Show that a map $f : A \to B$ is monic iff the following diagram is a pullback:

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, "id_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Dually show that a map $f : A \to B$ is epic iff the following diagram is a pushout:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "id_B"] \\ B \arrow[r, "id_B"'] & B \end{tikzcd}$$

Exercise 82. Consider the following diagram:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

Suppose that the right-hand square is a pullback. Show that the left-hand square is a pullback if and only if the outer rectangle is a pullback.

Exercise 83. Consider the following diagram:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

Suppose that the left-hand square is a pushout. Show that the right-hand square is a pushout if and only if the outer rectangle is a pushout.

Exercise 84. Show that if both squares

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & C \arrow[d, "i"] \\
B \arrow[r, "h"'] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "g'"'] & C' \arrow[d, "i'"] \\
B' \arrow[r, "h'"'] & D'
\end{tikzcd}
$$

are pushouts, then the square

$$
\begin{tikzcd}
A + A' \arrow[r, "f+f'"] \arrow[d, "g+g'"'] & C + C' \arrow[d, "i+i'"] \\
B + B' \arrow[r, "h+h'"'] & D + D'
\end{tikzcd}
$$

is a pushout.

Exercise 85. Show that in the category **Set** if both squares

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & C \arrow[d, "i"] \\
B \arrow[r, "h"'] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "g'"'] & C' \arrow[d, "i'"] \\
B' \arrow[r, "h'"'] & D'
\end{tikzcd}
$$

are pullbacks, then the square

$$
\begin{tikzcd}
A + A' \arrow[r, "f+f'"] \arrow[d, "g+g'"'] & C + C' \arrow[d, "i+i'"] \\
B + B' \arrow[r, "h+h'"'] & D + D'
\end{tikzcd}
$$

is also a pullback.