On the Logical Shadow of the Explicit Constructions

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"*It is equally stupid and simple to consider mathematics to be just an axiom system as it is to see a tree as nothing but a quantity of planks.*" L.E.J. Brouwer

In the intuitionistic tradition, mathematics has been considered as an incomplete story of our mental constructions and logic as the collection of the story's universal laws is nothing but a distorted incomplete shadow of the real mathematics. This role is clearly far from the foundational role that logic is usually believed to play. In the work we present here, we try to address this Brouwerian extrinsic interpretation of logic.

To formalize this interpretation, we have to first formalize the following two ingredients. First, the *constructions* that mathematics is supposed to be based on and then the *interpretation* that translates the logical formulas into the realm of the previously fixed constructions. For the former, there are many reasonable choices to make, including the computable functions formalized inside the standard model or HA, the set-theoretical functions in IZF or CZF, the terms in Martin Löf type theory or the morphisms in some strong enough categories such as locally Cartesian closed categories or toposes. In this talk and for the sake of simplicity, we set the functions in IZF as our fixed notion of construction. For the interpretations, though, we apparently have no choice but the canonical candidate of the BHK interpretation. However, we believe that the BHK interpretation is not a singular specific interpretation, but a name for a spectrum of different interpretations leading to different logics. Let us explain more, by introducing the two ends of the spectrum: The Heyting and the Brouwer interpretations:

Definition 1. *A Heyting interpretation is a map that assigns two sets $[A]_0$ and $[A]_1$ to any propositional formula A, such that:*

- $[p]_1$ *and* $[\bot]_1$ *are inhabited,* $[p]_0 \subseteq [p]_1$*, for any atomic formula p and* $[\bot]_0 = \emptyset$*,*
- $[A \wedge B]_1 = [A]_1 \times [B]_1$ *and* $[A \wedge B]_0 = \{(x, y) \in [A \wedge B]_1 \mid x \in [A]_0 \wedge y \in [B]_0\}$*,*
- $[A \vee B]_1 = [A]_1 + [B]_1$ *and* $[A \vee B]_0 = \{(i, x) \in [A \vee B]_1 \mid (i = 0 \to x \in [A]_0) \wedge (i = 1 \to x \in [B]_0)\}$*,*
- $[A \to B]_1 = [B]_1^{[A]_1}$ *and* $[A \to B]_0 = \{f \in [A \to B]_1 \mid \forall x \in [A]_0\ f(x) \in [B]_0\}$*.*

The sets $[A]_0$ and $[A]_1$ informally refer to the sets of the actual and possible constructions for A, respectively. A Brouwer interpretation is defined exactly in the same way, except for the disjunction case that is defined by: $[A \vee B]_1 = \|[A]_1 + [B]_1\|$, where $\| - \|$ is the propositional truncation, i.e., $\|X\| = \{x \in \{0\} \mid \exists y \in X\}$ and $[A \vee B]_0 = \{x \in \{0\} \mid \exists y \in [A]_0 \vee \exists y \in [B]_0\}$.

Given a construction for a disjunction, Heyting interpretation provides the complete information of the proved disjunct and the construction used for that proof. On the polar opposite side, the Brouwer interpretation uses the propositional truncation to collapse all the possible information in the construction, except probably its mere existence and hence no non-trivial information remained in a proof of a disjunction in this interpretation. This difference in the disjunction case is where the aforementioned spectrum enters the scene. Briefly, based on different amount of information that we assume a construction of a disjunction stores, we can develop different BHK interpretations.

Standing anywhere in the mentioned spectrum, it is also possible to restrict ourselves to a subclass of the interpretations to see how different conditions on the constructions lead to different logics. To have some examples, let us introduce the following classes of interpretations. An interpretation is called:

- **Markov**, if $\neg\neg\exists x \in [p]_0 \to \exists x \in [p]_0$, for any atomic formula p,
- **Kolmogorov**, if $[p]_1$ is an external finite set and $\neg\neg(x \in [p]_0) \to (x \in [p]_0)$, for any atomic formula p,
- **Proof-irrelevant**, if the condition that $[p]_0$ is inhabited implies $[p]_0 = [p]_1$, for any atomic formula p, i.e., if p has an actual proof, then all of its possible proofs are actual.

With the appropriate notions of construction and interpretation, we are ready to formalize what we mean by the theory and the logic of a calculus of constructions:

Definition 2. *Let $\mathcal{C}$ be a definable class of Heyting interpretations. By the $\mathcal{C}$-Heyting theory of* IZF*, denoted by $\mathbf{T}^H_{\mathcal{C}}(\mathsf{IZF})$, we mean the set of all propositional formulas A such that $\mathsf{IZF} \vdash \forall[-] \in \mathcal{C}\ \exists x \in [A]_0$, and by $\mathbf{L}^H_{\mathcal{C}}(\mathsf{IZF})$, we mean the set of all propositional formulas A such that $\sigma(A) \in \mathbf{T}^H_{\mathcal{C}}(\mathsf{IZF})$, for any propositional substitution σ. Similarly, define $\mathcal{C}$-Brouwer theory and logic of* IZF*, denoted by $\mathbf{T}^B_{\mathcal{C}}(\mathsf{IZF})$ and $\mathbf{L}^B_{\mathcal{C}}(\mathsf{IZF})$, respectively.*

Theorem 3. *Using M, K and PI to refer to the classes of Markov, Kolmogorov and proof-irrelevant interpretations, respectively, we have:*

- *(Brouwerian Constructivism)* $\mathbf{T}^B(\mathsf{IZF}) = \mathbf{T}^B_{PI}(\mathsf{IZF}) = \mathbf{L}^B_{MPI}(\mathsf{IZF}) = \mathbf{L}^B_K(\mathsf{IZF}) = \mathsf{IPC}$ *and* $\mathbf{T}^B_{MPI}(\mathsf{IZF}) = \mathsf{IPC} + \{\neg\neg p \to p \mid p \text{ is an atom}\}$.
- *(Heyting's Constructivism)* $\mathbf{T}^H(\mathsf{IZF}) \supseteq \mathsf{KP}$, *where* $\mathsf{KP} = \mathsf{IPC} + (\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$. *Therefore,* $\mathbf{T}^H(\mathsf{IZF}) \neq \mathsf{IPC}$ *and* $\mathbf{T}^H_{PI}(\mathsf{IZF}) = \mathsf{INP}$, *where* $\mathsf{INP} = \mathsf{IPC} + (A \to B \vee C) \to (A \to B) \vee (A \to C)$ *for any $\vee$-free formula A.*
- *(Russian Constructivism)* $\mathbf{L}^H_{MPI}(\mathsf{IZF}) = \mathbf{L}^H_K(\mathsf{IZF}) = \mathsf{ML}$, *where* ML *is Medvedev logic and* $\mathbf{T}^H_{MPI}(\mathsf{IZF}) = \mathsf{KP} + \{\neg\neg p \to p \mid p \text{ is an atom}\}$.