Uniform Lyndon interpolation for basic non-normal modal logics*

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Abstract

In this paper, a proof-theoretic method to prove uniform Lyndon interpolation for non-normal modal logics is introduced and applied to show that the logics E, M, MC, EN, MN have that property. In particular, these logics have uniform interpolation. Although for some of them the latter is known, the fact that they have uniform Lyndon interpolation is new. Also, the proof-theoretic proofs of these facts are new, as well as the constructive way to explicitly compute the interpolants that they provide. It is also shown that the non-normal modal logics EC and ECN do not have Craig interpolation, and whence no uniform (Lyndon) interpolation.

Keywords: non-normal modal logics, uniform interpolation, uniform Lyndon interpolation, Craig interpolation.

1 Introduction

Uniform interpolation, a strengthening of interpolation in which the interpolant only depends on the premise or the conclusion of an implication, is an intriguing logical property. One of the reasons is that it is hard to predict which logic does have the property and which does not. Well-behaved logics like K and KD have it, but then, other well-known modal logics, such as K4, do not. Early results on the subject were by Shavrukov [14], who proved uniform interpolation for the modal logic GL, and by Ghilardi [3] and Visser [16], who independently proved the same for K, followed later by Bílková, who showed that KT has the property as well [2]. Surprisingly, K4 and S4 do not have uniform interpolation, although they do have interpolation [2, 3]. Pitts provided the first proof-theoretic proof of uniform interpolation, for intuitionistic propositional logic, IPC, the smallest intermediate logic [11]. Results from [4, 8] imply that there are exactly seven intermediate logics with interpolation and that they are exactly the intermediate logics with uniform interpolation. Pitts' result is especially important to us, as also in our paper the approach is proof-theoretic.

The study of uniform interpolation in the context of non-normal modal logics has a more recent history. The area is less explored than its normal counterpart, but for several well-known non-normal logics uniform interpolation has been established, for example, for the monotone logic M [12], a result later extended in [10, 13]

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to other non-normal modal and conditional logics, such as E and basic conditional logic CK.

Our interest in the property of uniform interpolation lies in the fact that it can be used as a tool in what we would like to call *universal proof theory*, the area where one is concerned with the general behavior of proof systems, investigating problems such as the existence problem (when does a theory have a certain type of proof system?) and the equivalence problem (when are two proof systems equivalent?). The value of uniform interpolation for the existence problem has been addressed in a series of recent papers in which a method is developed to prove uniform interpolation that applies to many intermediate, (intuitionistic) modal, and substructural (modal) logics [1, 5, 6]. The proof-theoretic method makes use of proof systems for these logics, which in this case are sequent calculi, and shows that general conditions on the calculi imply uniform interpolation for the corresponding logic. Thus implying that any logic without uniform interpolation cannot have a sequent calculus satisfying these conditions. The generality of the conditions, such as closure under weakening, makes this into a powerful tool, especially for those classes of logics in which uniform interpolation is rare, such as intermediate logics. Note that in principle other regular properties than uniform interpolation could be used in this method, as long as the property is sufficiently rare to be of use.

In this paper we do not focus on the connection with the existence problem as just described, but rather aim to show the flexibility and utility of our method to prove uniform interpolation by showing that it can be extended to (yet) another class of logics, namely the class of non-normal modal logics, that it is constructive and can be easily adapted to prove not only uniform interpolation but even uniform Lyndon interpolation. Uniform Lyndon interpolation is a strengthening of uniform interpolation in which the interpolant respects the polarity of propositional variables (a definition follows in the next section). It first occurred in [7], where it was shown that several normal modal logics, including K and KD, have that property. In this paper we show that the non-normal modal logics E, M, MC, EN, MN have uniform Lyndon interpolation and the interpolant can be constructed explicitly from the proof. In the last part of this paper we show that the non-normal modal logics EC and ECN do not have interpolation, and whence no uniform (Lyndon) interpolation either. This surprising fact makes EC and ECN potential candidates for our approach to the existence problem discussed above, but that we have to leave for another paper.

That the logics E and M have uniform interpolation has already been established in [10, 12], but that they have uniform Lyndon interpolation is, as far as we know, a new insight. However, we consider the proof-theoretic method to prove uniform interpolation for non-normal modal logics the main contribution of this paper, as until now such proofs have always been semantical in nature. In [13] the search for proof-theoretic techniques to prove uniform interpolation in the setting of non-normal modal logics is explicitly mentioned in the conclusion of that paper.

2 Preliminaries

Set $\mathcal{L} = \{\wedge, \vee, \to, \bot, \Box\}$ as the language of modal logics. We use $\top$ and $\neg A$ as abbreviations for $\bot \to \bot$ and $A \to \bot$, respectively, and write $\varphi \in \mathcal{L}$ to indicate that φ is a formula in the language $\mathcal{L}$. The *weight of a formula* is defined inductively by: $w(\bot) = w(p) = 0$, for any atomic p and $w(A \circ B) = w(A) + w(B) + 1$, for any $\circ \in \{\wedge, \vee, \to\}$, and $w(\Box A) = w(A) + 1$.

Definition 2.1. The sets of positive and negative variables of a formula $\varphi \in \mathcal{L}$, denoted by $V^+(\varphi)$ and $V^-(\varphi)$, respectively, are defined recursively by:

- $V^+(p) = \{p\}$, $V^-(p) = V^+(\top) = V^-(\top) = V^+(\bot) = V^-(\bot) = \varnothing$, for atom p,
- $V^+(\varphi \circ \psi) = V^+(\varphi) \cup V^+(\psi)$ and $V^-(\varphi \circ \psi) = V^-(\varphi) \cup V^-(\psi)$, for $\circ \in \{\wedge, \vee\}$,
- $V^+(\varphi \to \psi) = V^-(\varphi) \cup V^+(\psi)$ and $V^-(\varphi \to \psi) = V^+(\varphi) \cup V^-(\psi)$,
- $V^+(\Box\varphi) = V^+(\varphi)$ and $V^-(\Box\varphi) = V^-(\varphi)$.

Define $V(\varphi)$ as $V^+(\varphi) \cup V^-(\varphi)$. For an atomic formula p, a formula φ is called *p^+-free* (*p^--free*), if $p \notin V^+(\varphi)$ ($p \notin V^-(\varphi)$). It is called *p-free* if $p \notin V(\varphi)$. Note that a formula is p-free iff p occurs nowhere in it.

For the sake of brevity, when we want to refer to both $V^+(\varphi)$ and $V^-(\varphi)$, we use the notation $V^\dagger(\varphi)$ with the condition "for any $\dagger \in \{+, -\}$". If we want to refer to one of $V^+(\varphi)$ and $V^-(\varphi)$ and its dual, we write $V^*(\varphi)$ for one and $V^\star(\varphi)$ for the other. For instance, if we state that for any atomic formula p, any $* \in \{+, -\}$ and any p^*-free formula φ, there is a $p^\star$-free formula ψ such that $\varphi \vee \psi \in L$, we are actually stating that if φ is p^+-free, there is a p^--free ψ such that $\varphi \vee \psi \in L$ and if φ is p^--free, there is a p^+-free ψ such that $\varphi \vee \psi \in L$.

Definition 2.2. A *logic* L is a set of formulas in $\mathcal{L}$ extending the set of classical tautologies, CPC, and closed under substitution and modus ponens $\varphi, \varphi \to \psi \vdash \psi$.

Definition 2.3. A logic L has *Lyndon interpolation property* if for any formulas $\varphi, \psi \in \mathcal{L}$, there is a formula $\theta \in \mathcal{L}$ such that $V^\dagger(\theta) \subseteq V^\dagger(\varphi) \cap V^\dagger(\psi)$, for any $\dagger \in \{+, -\}$ and $L \vdash \varphi \to \theta$ and $L \vdash \theta \to \psi$. A logic has *Craig interpolation* if it has the above properties, omitting all the superscripts $\dagger \in \{+, -\}$, everywhere.

Definition 2.4. A logic L has *uniform Lyndon interpolation property* if for any formula $\varphi \in \mathcal{L}$, any atomic formula p, and any $* \in \{+, -\}$, there exist two p^*-free formulas, denoted by $\forall^* p\varphi$ and $\exists^* p\varphi$, such that $V^\dagger(\exists^* p\varphi) \subseteq V^\dagger(\varphi)$ and $V^\dagger(\forall^* p\varphi) \subseteq V^\dagger(\varphi)$, for any $\dagger \in \{+, -\}$ and

(*i*) $L \vdash \forall^* p\varphi \to \varphi$,

(*ii*) for any p^*-free formula ψ if $L \vdash \psi \to \varphi$ then $L \vdash \psi \to \forall^* p\varphi$,

(*iii*) $L \vdash \varphi \to \exists^* p\varphi$, and

(*iv*) for any p^*-free formula ψ if $L \vdash \varphi \to \psi$ then $L \vdash \exists^* p\varphi \to \psi$.

A logic has *uniform interpolation property* if it has all the above properties, omitting the superscripts $*, \dagger \in \{+, -\}$, everywhere.

$$\overline{\Gamma, p \Rightarrow p, \Delta} \qquad \overline{\Gamma, \bot \Rightarrow, \Delta}$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} L\wedge \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} R\wedge$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} L\vee \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta} R\vee$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} L\to \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta} R\to$$

Figure 1: The sequent calculus **G3cp**. In the axiom, p must be an atomic formula.

Theorem 2.5. *If a logic L has uniform Lyndon interpolation property, then it has both Lyndon and uniform interpolation properties.*

Proof. For uniform interpolation, set $\forall p\varphi = \forall^+ p \forall^- p\varphi$ and $\exists p\varphi = \exists^+ p \exists^- p\varphi$. We only prove the claim for $\forall p\varphi$, as the case for $\exists p\varphi$ is similar. For the variable condition, it is clear that $V^\dagger(\forall p\varphi) \subseteq V^\dagger(\varphi)$, for any $\dagger \in \{+, -\}$. Hence, we have $V(\forall p\varphi) \subseteq V(\varphi)$. Moreover, $\forall p\varphi$ is p-free. Because $\forall^- p\varphi$ is p^--free by definition and as $V^-(\forall p\varphi) \subseteq V^-(\forall^- p\varphi)$, the formula $\forall^+ p\forall^- p\varphi$ is also p^--free. As $\forall^+ p\forall^- p\varphi$ is p^+-free by definition, we have $p \notin V(\forall p\varphi) = V^+(\forall p\varphi) \cup V^-(\forall p\varphi)$. For condition (i) in Definition 2.4, as $L \vdash \forall^+ p\forall^- p\varphi \to \forall^- p\varphi$ and $L \vdash \forall^- p\varphi \to \varphi$, we have $L \vdash \forall p\varphi \to \varphi$. For the condition (ii), if $L \vdash \psi \to \varphi$, for a p-free ψ, then ψ is also p^--free and hence $L \vdash \psi \to \forall^- p\varphi$. As ψ is also p^+-free, we have $L \vdash \psi \to \forall^+ p\forall^- p\varphi$.
For Lyndon interpolation, assume $L \vdash \varphi \to \psi$. For any $\dagger \in \{+, -\}$, set $P^\dagger = V^\dagger(\varphi) - [V^\dagger(\varphi) \cap V^\dagger(\psi)]$. Define $\theta = \exists^+ P^+ \exists^- P^- \varphi$, where by $\exists^\dagger\{p_1, \ldots, p_n\}^\dagger$ we mean $\exists p_1^\dagger \ldots \exists p_n^\dagger$. For the variable condition, since θ is $p^\dagger$-free for any $p \in P^\dagger$ and any $\dagger \in \{+, -\}$, we have $V^\dagger(\theta) \subseteq V^\dagger(\varphi) - P^\dagger \subseteq V^\dagger(\varphi) \cap V^\dagger(\psi)$. For the provability condition, it is clear that $L \vdash \varphi \to \theta$ and as ψ is $p^\dagger$-free for any $p \in P^\dagger$, we have $L \vdash \theta \to \psi$. □

2.1 Sequent calculi

We use capital Greek letters and the bar notation in $\bar{\varphi}$ and $\bar{C}$ to denote multisets. A *sequent* is an expression in the form $\Gamma \Rightarrow \Delta$, where Γ (the *antecedent*) and Δ (the *succedent*) are multisets of formulas. It is interpreted as $\bigwedge \Gamma \to \bigvee \Delta$. For sequents $S = (\Gamma \Rightarrow \Delta)$ and $T = (\Pi \Rightarrow \Lambda)$ we denote the sequent $\Gamma, \Pi \Rightarrow \Delta, \Lambda$ by $S \cdot T$, and the multisets Γ and Δ by S^a and S^s, respectively. Define $V^+(S) = V^-(S^a) \cup V^+(S^s)$ and $V^-(S) = V^+(S^a) \cup V^-(S^s)$ and the *weight of a sequent* as the sum of the weights of the formulas occurring in that sequent. A sequent S is *lower than* a sequent T, if the weight of S is less than the weight of T.

In this paper we are interested in modal extensions of the well-known sequent calculus **G3cp** from [15] (Figure 1) for classical logic CPC and its extension by the following two weakening rules, denoted by **G3W**:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} Lw \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta} Rw$$

$$\frac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\Box\varphi \Rightarrow \Box\psi} E \qquad \frac{\varphi_1, \cdots, \varphi_n \Rightarrow \psi \qquad \psi \Rightarrow \varphi_1 \qquad \cdots \qquad \psi \Rightarrow \varphi_n}{\Sigma, \Box\varphi_1, \cdots, \Box\varphi_n \Rightarrow \Box\psi, \Lambda} EC$$

$$\frac{\varphi \Rightarrow \psi}{\Box\varphi \Rightarrow \Box\psi} M \qquad \frac{\Rightarrow \psi}{\Rightarrow \Box\psi} N \qquad \frac{\Rightarrow \psi}{\Sigma \Rightarrow \Box\psi, \Lambda} NW \qquad \frac{\varphi_1, \cdots, \varphi_n \Rightarrow \psi}{\Box\varphi_1, \cdots, \Box\varphi_n \Rightarrow \Box\psi} MC$$

Figure 2: The modal rules

In each rule in **G3W**, the formulas outside $\Gamma \cup \Delta$ are called the *active formulas* of the rule and the only formula in the conclusion outside $\Gamma \cup \Delta$ is called the *main formula*. If S is the conclusion of an instance of a rule R, we say that R is *backwards applicable* to S. The modal rules by which we extend **G3cp** or **G3W** are given in Figure 2. For any such rule (X) except (EC), (N) and (NW), if we add it to **G3W** we denote the resulting system by **GX**, and if we add (N) to that system we get **GXN**. Note that **GMCN** is the usual system for the modal logic K. Moreover, if we add the rule (EC) to **G3cp**, we get **GEC** and if we also add the rule (NW), we get the system **GECN**. Note that the systems **GEC** and **GECN** have no explicit weakening rules.

The systems **GEC** and **GECN** are introduced in [9]. The others are equivalent to the systems introduced in [9]. The only difference is that in our representation, the weakening rules are explicitly present, while the extra context in the conclusion of the modal rules are omitted. We will present the systems as such for convenience in our later proofs. As the systems **GE**, **GM**, **GMC**, **GEN** and **GMN** are equivalent to the systems introduced in [9], they all admit the cut rule and the contraction rules. Moreover, the logics of these systems, i.e., the sets of formulas φ for which the systems prove $(\Rightarrow \varphi)$ are the well-known basic non-normal modal logics E, M, MC, EN and MN, respectively. The logics of the systems **GEC** and **GECN** are the logics EC and ECN, respectively [9].

Here are some remarks about the rules introduced above. First, for any rule, the weight of each of its premises is less than the weight of its conclusion. More specifically, note that the weight of $\Gamma, \Sigma \Rightarrow \Delta, \Lambda$ is less than the weight of $\Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Lambda$, as long as $\Gamma \cup \Delta$ is non-empty. Second, in any rule in **G3W**, if we add a multiset, both to the antecedent (succedent) of the premises and to the antecedent (succedent) of the conclusion, the result remains an instance of the rule. For the future reference, we call this property the *context extension property*. Conversely, if a multiset occurs both in the antecedent (succedent) of the premises and in the antecedent (succedent) of the conclusion and it does not contain any of the active formulas of the rule, then if we eliminate this multiset both from the premises and the conclusion, the result remains an instance of the same rule. We call this property the *context restriction property*. Third, for any rule in **G3W** and any $* \in \{+, -\}$, if the main formula φ is in the antecedent, then for any active formula α in the antecedent of a premise and any active formula β in the succedent of a premise, we have $V^*(\alpha) \cup V^\star(\beta) \subseteq V^*(\varphi)$, and if φ is in the succedent, we have $V^\star(\alpha) \cup V^*(\beta) \subseteq V^*(\varphi)$ (note the use of $*$ and $\star$). We call this property, the *variable preserving property*. As a consequence of this property for the rule $\dfrac{S_1 \ldots S_n}{S}$ in **G3W**, we have $\bigcup_{i=1}^{n} V^*(S_i) \subseteq V^*(S)$.

3 Uniform Lyndon Interpolation

In this section, we prove the uniform Lyndon interpolation property for the logics E, M, MC, EN, and MN. To this end, we need to first extend the notion to the sequent calculi of these logics. Since all these logics are classical, we only define the universal quantifier, as the existential quantifier is constructible by by the universal quantifier and negation.

Definition 3.1. Let G be one of the sequent calculi introduced in Preliminaries. The system G has *uniform Lyndon interpolation property* if for any sequent S, any atom p and any $* \in \{+, -\}$, there exists a formula $I^*_p(S)$ such that:

(var) $I^*_p(S)$ is p^*-free and $V^\dagger(I^*_p(S)) \subseteq V^\dagger(S)$, for any $\dagger \in \{+, -\}$,

(i) $S \cdot (I^*_p(S) \Rightarrow)$ is derivable in G,

(ii) for any sequent $\Gamma \Rightarrow \Delta$ such that $p \notin V^\star(\Gamma \Rightarrow \Delta)$, if $S \cdot (\Gamma \Rightarrow \Delta)$ is derivable in G then $(\Gamma \Rightarrow I^*_p(S), \Delta)$ is derivable in G.

$I^*_p(S)$ is called a *uniform* $\forall^*_p$*-interpolant of* S in G. For any set of rules $\mathcal{R}$ of G, a formula $I^*_{p,\mathcal{R}}(S)$ is called a *uniform* $\forall^*_p$*-interpolant of* S *with respect to* $\mathcal{R}$, if it satisfies the conditions (var) and (i), when $I^*_p(S)$ is replaced by $I^*_{p,\mathcal{R}}(S)$, and:

(ii') for any sequent $\Gamma \Rightarrow \Delta$ such that $p \notin V^\star(\Gamma \Rightarrow \Delta)$, if there is a derivation of $S \cdot (\Gamma \Rightarrow \Delta)$ in G whose last inference rule is an instance of a rule in $\mathcal{R}$, then $(\Gamma \Rightarrow I^*_{p,\mathcal{R}}(S), \Delta)$ is derivable in G.

The following theorem connects uniform Lyndon interpolation property for sequent calculi to the original version.

Theorem 3.2. *Let G be one of the sequent calculi introduced in Preliminaries and L be its logic. Then, G has uniform Lyndon interpolation property iff L has uniform Lyndon interpolation property.*

Proof. If G has uniform Lyndon interpolation, then set $\forall^* pA = I^*_p(\Rightarrow A)$ and $\exists^* pA = \neg\forall^\star p \neg A$. It is easy to see that these two formulas play the role of the uniform Lyndon interpolants. Conversely, if L has uniform Lyndon interpolation, then set $I^*_p(\Gamma \Rightarrow \Delta) = \forall^* p(\bigwedge \Gamma \to \bigvee \Delta)$. It is again easy to see that $I^*_p(\Gamma \Rightarrow \Delta)$ has the required properties. □

Our strategy to prove uniform Lyndon interpolation for the logics E, M, MC, EN, and MN is to prove the same property for their sequent calculi. From now on, up to Subsection 3.1, we assume that G is one of the calculi **GE**, **GM**, **GMC**, **GEN**, and **GMN**. As stated previously, backward applications of the rules decreases the weight of the sequent. Using this property and recursion on the weight of the sequents, for any given sequent $S = (\Gamma \Rightarrow \Delta)$, any atom p and any $* \in \{+, -\}$, we first define a p^*-free formula $\forall^* pS$ and then by induction on the weight of S, we prove that $\forall^* pS$ meets the conditions in Definition 3.1. Towards that end, both in the definition of $\forall^* pS$ and in the proof of its properties, we must address all the rules of the system G, one by one. To make the presentation uniform, modular, and more clear, we divide the rules of G into two families: the rules of **G3W** and the modal rules specific for G. The rules in the first class has one of the following forms:

$$\frac{\{\Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \Delta\}_i}{\Gamma, \varphi \Rightarrow \Delta} \qquad \frac{\{\Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \Delta\}_i}{\Gamma \Rightarrow \varphi, \Delta}$$

where Γ and Δ are free for all multiset substitutions, and $\bar{\varphi}_i$'s and $\bar{\psi}_i$'s are multisets of formulas (possibly empty). The rules have the variable preserving condition, i.e., given $* \in \{+, -\}$, for the left rule $\bigcup_i \bigcup_{\theta \in \bar{\varphi}_i} V^*(\theta) \cup \bigcup_i \bigcup_{\theta \in \bar{\psi}_i} V^\star(\theta) \subseteq V^*(\varphi)$, and for the right one $\bigcup_i \bigcup_{\theta \in \bar{\varphi}_i} V^\star(\theta) \cup \bigcup_i \bigcup_{\theta \in \bar{\psi}_i} V^*(\theta) \subseteq V^*(\varphi)$.
Rather than addressing each rule in **G3W**, we simply address these two forms that cover all the rules in **G3W**.

Lemma 3.3. *For any sequent S, any atomic formula p and any $* \in \{+, -\}$, a uniform $\forall_p^*$-interpolant of S with respect to the set of all axioms of G exists.*

Proof. Let us define a formula $\forall_{ax}^* pS$: if S is provable, define it as $\top$, otherwise, define $\forall_{ax}^* pS$ as the disjunction of all p^*-free formulas in S^s and the negation of all $p^\star$-free formulas in S^a. We show that $\forall_{ax}^* pS$ is the uniform $\forall_p^*$-interpolant of S with respect to the set of axioms of G. It is easy to see that $\forall_{ax}^* pS$ is p^*-free, $V^\dagger(\forall_{ax}^* pS) \subseteq V^\dagger(S)$, for any $\dagger \in \{+, -\}$ and $S \cdot (\forall_{ax}^* pS \Rightarrow)$ is provable in G. To prove the condition (ii') in Definition 3.1, if S is provable, then as $\forall_{ax}^* pS = \top$, we have $\bar{C} \Rightarrow \forall_{ax}^* pS, \bar{D}$. If S is not provable, then let $S \cdot (\bar{C} \Rightarrow \bar{D})$ be an axiom. There are two cases to consider. First, if $S \cdot (\bar{C} \Rightarrow \bar{D})$ is in the form $\Gamma, q \Rightarrow q, \Delta$, where q is an atomic formula. Then, if $q \notin \bar{C}$ and $q \notin \bar{D}$, we have $q \in \Gamma \cap \Delta$ and hence the sequent S is provable which contradicts our assumption. Therefore, either $q \in \bar{C}$ or $q \in \bar{D}$. If $q \in \bar{C} \cap \bar{D}$, then $\bar{C} \Rightarrow \forall_{ax}^* pS, \bar{D}$ is provable. Hence, we assume either $q \in \bar{C}$ and $q \notin \bar{D}$ or $q \notin \bar{C}$ and $q \in \bar{D}$. In the first case, if $q \in \bar{C}$, it is p^*-free and since it occurs in Δ, it is a disjunct in $\forall_{ax}^* pS$. Hence, $\bar{C} \Rightarrow \forall_{ax}^* pS, \bar{D}$ is provable. In the second case, if $q \in \bar{D}$, it is $p^\star$-free and as $q \in \Gamma$, its negation occurs in $\forall_{ax}^* pS$. Therefore $\bar{C} \Rightarrow \forall_{ax}^* pS, \bar{D}$ is provable.
If $S \cdot (\bar{C} \Rightarrow \bar{D})$ is in the form $\Gamma, \bot \Rightarrow \Delta$, then $\bot \in \bar{C}$, because otherwise, $\bot \in \Gamma$ and hence S will be provable. Now, since $\bot \in \bar{C}$, we have $\bar{C} \Rightarrow \forall_{ax}^* pS, \bar{D}$. □

Definition 3.4. Let $\mathcal{U}_p^*(S)$ be the statement that "all sequents lower than S have uniform $\forall_p^*$-interpolants." A calculus G has *MUIP* if for any sequent S, any atomic formula p, and any $* \in \{+, -\}$, there exists a formula $\forall_m^* pS$ such that if $\mathcal{U}_p^*(S)$, then $\forall_m^* pS$ is a uniform $\forall_p^*$-interpolant for S with respect to the set of modal rules of G.

Theorem 3.5. *If a sequent calculus G has MUIP, then it has uniform Lyndon interpolation property.*

Proof. Define a formula $\forall^* pS$ by recursion on the weight of S: if S is provable define it as $\top$, otherwise, define $\forall^* pS$ as:

$$\bigvee_R \Big(\bigwedge_i \forall^* pS_i\Big) \vee (\forall_{ax}^* pS) \vee (\forall_m^* pS)$$

where the first disjunction is over all rules R in **G3W** backward applicable to S, where S is the consequence and S_i's are the premises. The second disjunct, $\forall_{ax}^* pS$, is a uniform $\forall_p^*$-interpolant of S with respect to the set of axioms of G that Lemma 3.3 provides. The third disjunct, $\forall_m^* pS$, is the formula that MUIP provides. To prove that the formula $\forall^* pS$ is a $\forall_p^*$-interpolant for S, we use induction on the weight of S to prove:

(var) $\forall^* pS$ is p^*-free and $V^\dagger(\forall^* pS) \subseteq V^\dagger(S)$, for any $\dagger \in \{+, -\}$,

(i) $S \cdot (\forall^* pS \Rightarrow)$ is provable in G,

(ii) for any $p^\star$-free sequent $\bar{C} \Rightarrow \bar{D}$, if $S \cdot (\bar{C} \Rightarrow \bar{D})$ is derivable in G then $\bar{C} \Rightarrow \forall^* pS, \bar{D}$ is derivable in G.

By induction hypothesis, (var), (i), and (ii) hold for all sequents T lower than S. Now, (var) also holds for $\forall^* pS$, because both $\forall^*_{ax} pS$ and $\forall^*_m pS$ satisfy (var) and all rules in **G3W** have the variable preserving property.
To prove (i), it is enough to show that the following are provable in G:

$$S \cdot (\bigwedge_i \forall^* pS_i \Rightarrow) \quad (1)\ , \quad S \cdot (\forall^*_{ax} pS \Rightarrow) \quad (2)\ , \quad S \cdot (\forall^*_m pS \Rightarrow) \quad (3).$$

Sequent (3) is provable by induction hypothesis and the assumption that G has MUIP. Sequent (2) is proved in Lemma 3.3. For the sequent (1), assume that the rule R of **G3W** is backward applicable to S, i.e., the premises of R are S_i's and its conclusion S. As S_i's are lower than S, by induction hypothesis we have $S_i \cdot (\forall^* pS_i \Rightarrow)$. Therefore, by weakening, we have $S_i \cdot (\{\forall^* pS_i\}_i \Rightarrow)$. Since any rule in **G3W** has the context extension property, we can add $\{\forall^* pS_i\}_i$ to the antecedent of both premises and conclusion and by the rule itself, we have $S \cdot (\{\forall^* pS_i\}_i \Rightarrow)$ and hence $S \cdot (\bigwedge_i \forall^* pS_i \Rightarrow)$.
For (ii), we use another induction on the length of the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$. Let $S \cdot (\bar{C} \Rightarrow \bar{D})$ be derivable in G. If it is an axiom, we have $\bar{C} \Rightarrow \bar{D}, \forall^*_{ax} pS$ by Lemma 3.3, and hence $\bar{C} \Rightarrow \bar{D}, \forall^* pS$. If the last rule is a rule in **G3W** of the form:

$$\frac{\{\Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \Delta\}_i}{\Gamma, \varphi \Rightarrow \Delta}\ ,$$

then there are two cases to consider, i.e., either $\varphi \in \bar{C}$ or $\varphi \in S^a$. If $\varphi \in \bar{C}$, then set $\bar{C}' = \bar{C} - \{\varphi\}$. Since $\varphi \in \bar{C}$, it is p^*-free by the assumption and φ_i's are all p^*-free and ψ_i's are all $p^\star$-free by the variable preserving property. By induction hypothesis, as $(\bar{C}', \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \bar{D})$ is $p^\star$-free and $S \cdot (\bar{C}', \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \bar{D})$ has a shorter proof, we have $\bar{C}', \bar{\varphi}_i \Rightarrow \forall^* pS, \bar{\psi}_i, \bar{D}$. By using the rule itself, we have

$$\frac{\{\bar{C}', \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \forall^* pS, \bar{D}\}_i}{\bar{C}', \varphi \Rightarrow \forall^* pS, \bar{D}}$$

which implies $\bar{C} \Rightarrow \forall^* pS, \bar{D}$.
If $\varphi \notin \bar{C}$, then both $\bar{C}$ and $\bar{D}$ do not contain any active formula of the rule and hence the last rule is in form:

$$\frac{\{\bar{C}, \Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \bar{D}, \Delta\}_i}{\bar{C}, \Gamma, \varphi \Rightarrow \bar{D}, \Delta}\ .$$

By context restriction property, if we erase $\bar{C}$ and $\bar{D}$ both on the premises and the consequence of the last rule, the rule remains valid and it changes to:

$$\frac{\{\Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \Delta\}_i}{\Gamma, \varphi \Rightarrow \Delta}\ .$$

Therefore, the rule is backward applicable to the sequent $S = (\Gamma, \varphi \Rightarrow \Delta)$. Set $S_i = (\Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \Delta)$. As the weight of S_i's are less than the weight of S and $S_i \cdot (\bar{C} \Rightarrow \bar{D})$ are provable, by induction hypothesis, we have $\bar{C} \Rightarrow \forall^* p S_i, \bar{D}$. Hence, $\bar{C} \Rightarrow \bigwedge_i \forall^* p S_i, \bar{D}$ and as $\bigwedge_i \forall^* p S_i$ is a disjunct in $\forall^* p S$, we have $\bar{C} \Rightarrow \forall^* p S, \bar{D}$. The case where the last rule is in **GW3** with its main formula in the antecedent is similar. For the modal rules, by induction hypothesis $\mathcal{U}_p^*(S)$ and the assumption that G has MUIP, we get that $\forall_m^* p S$ is a uniform $\forall_p^*$-interpolant for S with respect to the set of all modal rules of G. By (ii') in Definition 3.1, this gives $\bar{C} \Rightarrow \forall_m^* p S, \bar{D}$ and hence $\bar{C} \Rightarrow \forall^* p S, \bar{D}$. □

In the upcoming subsections, for the following choices of the system G, we show that it has MUIP. Therefore, by Theorem 3.5 and Theorem 3.2, we will have:

Theorem 3.6. *Logics* E*,* M*,* MC*,* EN *and* MN *have uniform Lyndon interpolation property and hence both uniform interpolation and Lyndon interpolation properties.*

3.1 Modal Logics M and MN

We only discuss the cases for M and MN. Let G be either **GM** or **GMN**. We will show that G has MUIP. To define $\forall_m^* p S$, if $\neg \mathcal{U}_p^*(S)$, define $\forall_m^* p S$ as $\bot$. If $\mathcal{U}_p^*(S)$, (i.e., for any sequent T lower than S a uniform $\forall_p^*$-interpolant, denoted by $\forall^* p T$, exists), define $\forall_m^* p S$ in the following way: if S is provable, define $\forall_m^* p S$ as $\top$, otherwise, if it is of the form $(\Box \varphi \Rightarrow)$, define $\forall_m^* p S = \neg \Box \neg \forall^* p S'$, where $S' = (\varphi \Rightarrow)$, if S is of the form $(\Rightarrow \Box \psi)$, define $\forall_m^* p S = \Box \forall^* p S''$, where $S'' = (\Rightarrow \psi)$, and otherwise, define $\forall_m^* p S = \bot$. Note that $\forall_m^* p S$ is well-defined as we have $\mathcal{U}_p^*(S)$ and S' in the first case and S'' in the second case are lower than S.
To show that G has MUIP, we assume $\mathcal{U}_p^*(S)$ to prove the three conditions (var), (i) and (ii') in Definition 3.1 for $\forall_m^* p S$. First, note that using $\mathcal{U}_p^*(S)$ on $(\varphi \Rightarrow)$ and $(\Rightarrow \psi)$ that are lower than $(\Box \varphi \Rightarrow)$ and $(\Rightarrow \Box \psi)$, respectively, the variable conditions are implied from (var) for S' and S'', respectively.
For (i), if S is provable, there is nothing to prove. Otherwise, if $S = (\Box \varphi \Rightarrow)$ then $\forall_m^* p S = \neg \Box \neg \forall^* p S'$, where $S' = (\varphi \Rightarrow)$. As S' is lower than S, we have $(\varphi, \forall^* p S' \Rightarrow)$ by $\mathcal{U}_p^*(S)$, which implies $(\varphi \Rightarrow \neg \forall^* p S')$. Using the rule (M), we get $(\Box \varphi \Rightarrow \Box \neg \forall^* p S')$, which is equivalent to $(\Box \varphi, \neg \Box \neg \forall^* p S' \Rightarrow)$. Hence, $S \cdot (\forall_m^* p S \Rightarrow)$ is provable.
If S is not provable, $S = (\Rightarrow \Box \psi)$ and $S'' = (\Rightarrow \psi)$, we have $\forall_m^* p S = \Box \forall^* p S''$. Using $\mathcal{U}_p^*(S)$ on S'' and the fact that S'' is lower than S, we have $(\forall^* p S'' \Rightarrow \psi)$ and by the rule (M), we can show that $S \cdot (\Box \forall^* p S'' \Rightarrow)$ is provable in G. If S is not provable and has none of the mentioned forms, as $\forall_m^* p S = \bot$, there is nothing to prove.
For (ii'), let $S \cdot (\bar{C} \Rightarrow \bar{D})$ be derivable in G for a $p^\star$-free sequent $\bar{C} \Rightarrow \bar{D}$ and the last rule is a modal rule. We want to show that $\bar{C} \Rightarrow \forall_m^* p S, \bar{D}$ is derivable in G. If the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is (M), the sequent must have the form $(\Box \varphi \Rightarrow \Box \psi)$ and the rule must be in form:

$$\frac{\varphi \Rightarrow \psi}{\Box \varphi \Rightarrow \Box \psi} M$$

If S is provable, as $\forall_m^* p S = \top$, we clearly have $\bar{C} \Rightarrow \forall_m^* p S, \bar{D}$. Assume S is not provable and hence $\bar{C} \cup \bar{D}$ cannot be empty. Therefore, there are three cases to

consider, either $\bar{C}$ is $\Box\varphi$ or $\bar{D}$ is $\Box\psi$ or both. First, if $\bar{C} = \Box\varphi$ and $\bar{D} = \varnothing$, then, S is of the form $S = (\Rightarrow \Box\psi)$ and φ is p^*-free. Set $S'' = (\Rightarrow \psi)$. Then $\forall^*_m pS = \Box\forall pS''$. As S'' is lower than S, by $\mathcal{U}^*_p(S)$ we have $(\varphi \Rightarrow \forall^* pS'')$. Using the modal rule (M), we have $(\Box\varphi \Rightarrow \Box\forall^* pS'')$ and hence $(\bar{C} \Rightarrow \forall^*_m pS, \bar{D})$.
In the second case, assume $\bar{C} = \varnothing$ and $\bar{D} = \Box\psi$. Hence, $S = (\Box\varphi \Rightarrow)$ and ψ is $p^\star$-free. Set $S' = (\varphi \Rightarrow)$. Hence, $\forall^*_m pS = \neg\Box\neg\forall^* pS'$. Since $(\varphi \Rightarrow \psi)$ is provable in G and S' is lower than S, by $\mathcal{U}^*_p(S)$ we have $(\Rightarrow \forall^* pS', \psi)$, or equivalently $(\neg\forall^* pS' \Rightarrow \psi)$. Using the modal rule (M), we get $(\Box\neg\forall^* pS' \Rightarrow \Box\psi)$ or equivalently $(\Rightarrow \neg\Box\neg\forall^* pS', \Box\psi)$. Therefore, we have $(\Rightarrow \forall^*_m pS, \Box\psi)$ or $(\bar{C} \Rightarrow \forall^*_m pS, \bar{D})$.
In the third case, if $\bar{C} = \Box\varphi$ and $\bar{D} = \Box\psi$, then S is the empty sequent and $\bar{C} \Rightarrow \bar{D}$ is provable. Hence, $\bar{C} \Rightarrow \forall^*_m pS, \bar{D}$ is also provable.
For the case $G = \mathbf{GMN}$, if $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Rightarrow \Box\psi)$ is proved by the rule (N), it must have the following form:

$$\frac{\Rightarrow \psi}{\Rightarrow \Box\psi}\, N$$

Then $\bar{C} = \varnothing$ and there are two cases to consider. The first case is when $S = (\Rightarrow \Box\psi)$ and $\bar{D} = \varnothing$. Then, it means that S is provable which contradicts our assumption. The second case is when $S = (\Rightarrow)$ and $\bar{D} = \Box\psi$. Hence, $\bar{C} \Rightarrow \bar{D}$ is provable and we have the provability of $\bar{C} \Rightarrow \forall^*_m pS, \bar{D}$ in G.

3.2 Modal Logic MC

Similar to the argument of the previous subsection, to define $\forall^*_m pS$, if $\neg\mathcal{U}^*_p(S)$, define $\forall^*_m pS$ as $\bot$. If $\mathcal{U}^*_p(S)$, (i.e., for any sequent T lower than S the uniform $\forall^*_p$-interpolant, denoted by $\forall^* pT$, exists), define $\forall^*_m pS$ as the following: if S is provable, define $\forall^*_m pS = \top$. Otherwise, if S is of the form $(\Box\varphi_1, \cdots, \Box\varphi_i \Rightarrow)$, for some $i \geqslant 1$, define $\forall^*_m pS = \neg\Box\neg\forall^* pS'$, where $S' = (\varphi_1, \cdots, \varphi_i \Rightarrow)$. If S is of the form $(\Rightarrow \Box\psi)$, define $\forall^*_m pS = \Box\forall^* pS''$, where $S'' = (\Rightarrow \psi)$. If S is of the form $(\Box\varphi_1, \cdots, \Box\varphi_i \Rightarrow \Box\psi)$, for some $i \geqslant 1$, define $\forall^*_m pS = \Box\forall^* pS''$, where $S'' = (\varphi_1, \cdots, \varphi_i \Rightarrow \psi)$. Otherwise, define $\forall^*_m pS = \bot$. Note that $\forall^*_m pS$ is well-defined as we assumed $\mathcal{U}^*_p(S)$ and in each case S' or S'' are lower than S.
To show that **GMC** has MUIP, we assume $\mathcal{U}^*_p(S)$ to prove the three conditions (var), (i) and (ii') in Definition 3.1 for $\forall^*_m pS$. The condition (var) is an immediate consequence of $\mathcal{U}^*_p(S)$ and the fact that S' or S'' are lower than S. For (i), if S is provable, there is nothing to prove. If S is of the form $(\Box\varphi_1, \cdots, \Box\varphi_i \Rightarrow)$ and $\forall^*_m pS = \neg\Box\neg\forall^* pS'$, where $S' = (\varphi_1, \cdots, \varphi_i \Rightarrow)$, as S' is lower than S, by $\mathcal{U}^*_p(S)$ we have $(\varphi_1, \cdots, \varphi_i, \forall^* pS' \Rightarrow)$ or equivalently $(\varphi_1, \cdots, \varphi_i \Rightarrow \neg\forall^* pS')$. Using the rule (MC), we get $(\Box\varphi_1, \cdots, \Box\varphi_i \Rightarrow \Box\neg\forall^* pS')$, which is equivalent to $(\Box\varphi_1, \cdots, \Box\varphi_i, \neg\Box\neg\forall^* pS' \Rightarrow)$ and hence $S \cdot (\forall^*_m pS \Rightarrow)$.
If S is of the form $(\Rightarrow \Box\psi)$ and $S'' = (\Rightarrow \psi)$, or S is of the form $(\Box\varphi_1, \cdots, \Box\varphi_i \Rightarrow \Box\psi)$, for some $i \geqslant 1$ and S'' is of the form $(\varphi_1, \cdots, \varphi_i \Rightarrow \psi)$, we have $\forall^*_m pS = \Box\forall^* pS''$. In both cases, using $\mathcal{U}^*_p(S)$ on S'', we have either $\forall^* pS'' \Rightarrow \psi$ or the sequent $\varphi_1, \cdots, \varphi_i, \forall^* pS'' \Rightarrow \psi$, respectively. In both cases, using the rule (MC), we can show that $S \cdot (\Box\forall^* pS'' \Rightarrow)$ is provable and hence $S \cdot (\forall^*_m pS \Rightarrow)$.
For (ii'), let $S \cdot (\bar{C} \Rightarrow \bar{D})$ be derivable in **GMC** and the last rule is the modal rule (MC), for a $p^\star$-free sequent $\bar{C} \Rightarrow \bar{D}$. We want to show that $\bar{C} \Rightarrow \forall^*_m pS, \bar{D}$ is derivable in **GMC**. If S is provable, as $\forall^*_m pS = \top$, we have $\bar{C} \Rightarrow \forall^*_m pS, \bar{D}$.

Therefore, we assume that S is not provable. As the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is (MC), the sequent must have the form $(\Box\varphi_1, \cdots, \Box\varphi_n \Rightarrow \Box\psi)$ and the rule is:

$$\frac{\varphi_1, \cdots, \varphi_n \Rightarrow \psi}{\Box\varphi_1, \cdots, \Box\varphi_n \Rightarrow \Box\psi} MC$$

Then, there are two cases to consider, either $\bar{D} = \Box\psi$ or $\bar{D} = \varnothing$. First, assume S is of the form $(\Box\varphi_1, \cdots, \Box\varphi_i \Rightarrow)$, for $i \leqslant n$, then $\bar{C} = \Box\varphi_{i+1}, \cdots, \Box\varphi_n$ and $\bar{D} = \Box\psi$ and hence $\varphi_{i+1}, \cdots, \varphi_n \Rightarrow \psi$ is $p^\star$-free. Set $S' = (\varphi_1, \cdots, \varphi_i \Rightarrow)$. By the form of S, we have $\forall^*_m pS = \neg\Box\neg\forall^* pS'$. As S' is lower than S, by $\mathcal{U}^*_p(S)$, we have $(\varphi_{i+1}, \cdots, \varphi_n \Rightarrow \forall^* pS', \psi)$. Hence, by moving $\forall^* pS'$ to the left, applying the rule (MC) and moving back, we have $(\Box\varphi_{i+1}, \cdots, \Box\varphi_n \Rightarrow \neg\Box\neg\forall^* pS', \Box\psi)$ or equivalently $(\bar{C} \Rightarrow \forall^*_m pS, \bar{D})$.
If S is of the form $\Box\varphi_1, \cdots, \Box\varphi_i \Rightarrow \Box\psi$, for some $i \leqslant n$, we must have $\bar{C} = \Box\varphi_{i+1}, \cdots, \Box\varphi_n$ and $\bar{D} = \varnothing$. Hence, $\varphi_{i+1}, \cdots, \varphi_n$ are p^*-free. Note that $i < n$, because if $i = n$, then S will be provable that contradicts our assumption. Set $S'' = (\varphi_1, \cdots, \varphi_i \Rightarrow \psi)$. As S'' is lower than S, by $\mathcal{U}^*_p(S)$ we have $\varphi_{i+1}, \cdots, \varphi_n \Rightarrow \forall^* pS''$. By the fact that $i < n$, we can apply the rule (MC) to prove $\Box\varphi_{i+1}, \cdots, \Box\varphi_n \Rightarrow \Box\forall^* pS''$ and hence $(\bar{C} \Rightarrow \forall^*_m pS, \bar{D})$.

3.3 Modal Logics E and EN

Let G be either **GE** or **GEN**. Similar to the argument of the previous subsections, to define $\forall^*_m pS$, if $\neg\mathcal{U}^*_p(S)$, define $\forall^*_m pS$ as $\bot$. If $\mathcal{U}^*_p(S)$, (i.e., for any sequent T lower than S the uniform $\forall^*_p$-interpolant, denoted by $\forall^* pT$, exists), define $\forall^*_m pS$ as the following: if S is provable in G, define $\forall^*_m pS = \top$. Otherwise, if it has the form $S = (\Box\varphi \Rightarrow)$ and the sequents $(\neg\forall^* pS' \Rightarrow \varphi)$ and $(\varphi \Rightarrow \neg\forall^* pS')$ are provable in G, for $S' = (\varphi \Rightarrow)$, define $\forall^*_m pS = \neg\Box\neg\forall^* pS'$. If S has the form $(\Rightarrow \Box\psi)$ and the sequents $(\forall^* pS'' \Rightarrow \psi)$ and $(\psi \Rightarrow \forall^* pS'')$ are provable in G, for $S'' = (\Rightarrow \psi)$, define $\forall^*_m pS = \Box\forall^* pS''$. Otherwise, define $\forall^*_m pS = \bot$. Note that $\forall^* pS$ is well-defined as in each case S' and S'' are lower than S and we assumed $\mathcal{U}^*_p(S)$.
To show that G has MUIP we assume $\mathcal{U}^*_p(S)$ to prove the three conditions (var), (i) and (ii') in Definition 3.1 for $\forall^*_m pS$. The condition (var) is a simple consequence of $\mathcal{U}^*_p(S)$ and the fact that S' or S'' are lower than S. For (i), if S is provable, there is nothing to prove. If $S = (\Box\varphi \Rightarrow)$ and $S' = (\varphi \Rightarrow)$ and the sequents $(\neg\forall^* pS' \Rightarrow \varphi)$ and $(\varphi \Rightarrow \neg\forall^* pS')$ are provable in G, then using the rule (E), we have $(\Box\varphi \Rightarrow \Box\neg\forall^* pS')$ which implies $(\Box\varphi, \neg\Box\neg\forall^* pS' \Rightarrow)$ and hence $S \cdot (\forall^*_m pS \Rightarrow)$ is provable in G.
If $S = (\Rightarrow \Box\psi)$ and $S'' = (\Rightarrow \psi)$ and the sequents $(\forall^* pS'' \Rightarrow \psi)$ and $(\psi \Rightarrow \forall^* pS'')$ are provable in G, then using the rule (E), we have $(\Box\forall^* pS'' \Rightarrow \Box\psi)$ and hence $S \cdot (\forall^*_m pS \Rightarrow)$ is provable in G. If $\forall^*_m pS = \bot$, there is nothing to prove.
For (ii'), if S is provable, then $\forall^*_m pS = \top$ and hence $\bar{C} \Rightarrow \forall^*_m pS, \bar{D}$. Therefore, assume that S is not provable. If the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is the rule (E), the sequent $S \cdot (\bar{C} \Rightarrow \bar{D})$ is of the form $\Box\varphi \Rightarrow \Box\psi$. There are four cases to consider based on if $\bar{C}$ or $\bar{D}$ are empty or not. First, if $\bar{C} = \bar{D} = \varnothing$, then S is provable which contradicts our assumption. If S is the empty sequent $(\Rightarrow)$, then $\bar{C} \Rightarrow \bar{D}$ is provable and hence $\bar{C} \Rightarrow \forall^*_m pS, \bar{D}$ is provable.
If S is of the form $S = (\Box\varphi \Rightarrow)$, then $\bar{C} = \varnothing$ and $\bar{D} = \Box\psi$ and hence ψ is $p^\star$-free. Set $S' = (\varphi \Rightarrow)$ and as the last rule is (E), both of the sequents $\varphi \Rightarrow \psi$

and $\psi \Rightarrow \varphi$ are provable. By $\mathcal{U}_p^*(S)$ and the fact that S' is lower than S, we have $(\varphi, \forall^* p S' \Rightarrow)$ or equivalently, $(\varphi \Rightarrow \neg\forall^* p S')$. Again by $\mathcal{U}_p^*(S)$ for S', the provability of $S' \cdot (\Rightarrow \bar{D}) = (\varphi \Rightarrow \psi)$ and the fact that $(\Rightarrow \psi)$ is $p^\star$-free, we have $(\Rightarrow \forall^* p S', \psi)$ or equivalently, $(\neg\forall^* p S' \Rightarrow \psi)$. Since $(\varphi \Rightarrow \psi)$ and $(\psi \Rightarrow \varphi)$ are provable, by cut we can prove the equivalence between φ, ψ and $\neg\forall^* p S'$. Using this fact, we have:

$$\frac{\psi \Rightarrow \neg\forall^* p S' \qquad \neg\forall^* p S' \Rightarrow \psi}{\Box\neg\forall^* p S' \Rightarrow \Box\psi} E$$

Hence, $(\Rightarrow \neg\Box\neg\forall^* p S', \Box\psi)$. Then, as S has the form $S = (\Box\varphi \Rightarrow)$ and both of the sequents $(\neg\forall^* p S' \Rightarrow \varphi)$ and $(\varphi \Rightarrow \neg\forall^* p S')$ are provable in G, by definition we have $\forall_m^* p S = \neg\Box\neg\forall^* p S'$ and hence $(\Rightarrow \neg\Box\neg\forall^* p S', \Box\psi) = (\bar{C} \Rightarrow \forall_m^* p S, \bar{D})$ is provable in G. The last case where S has the form $S = (\Rightarrow \Box\psi)$ and $\bar{C} = \Box\varphi$ and $\bar{D} = \varnothing$ is similar.
For the case $G = \mathbf{GEN}$, if $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Rightarrow \Box\psi)$ is proved by the rule (N), it must have the following form $\dfrac{\Rightarrow \psi}{\Rightarrow \Box\psi} N$. Then $\bar{C} = \varnothing$ and there are two cases to consider. The first case is when $S = (\Rightarrow \Box\psi)$ and $\bar{D} = \varnothing$, which means that S is provable which contradicts our assumption. The second case is when $S = (\Rightarrow)$ and $\bar{D} = \Box\psi$, and since in this case S is the empty sequent and hence $\bar{C} \Rightarrow \bar{D}$ is provable, we have the provability of $\bar{C} \Rightarrow \forall_m^* p S, \bar{D}$ in G.

4 Modal Logics EC and ECN

In this section we prove that the logics EC and ECN do not enjoy the Craig interpolation property. To this end, if L is either EC or ECN, we show that $L \vdash \Box(\neg q \wedge r) \rightarrow (\Box(p \wedge q) \rightarrow \Box\bot)$, where p, q, and r are three distinct atomic formulas while there is no formula θ such that $V(\theta) \subseteq \{q\}$ and both formulas $\Box(\neg q \wedge r) \rightarrow \theta$ and $\theta \rightarrow (\Box(p \wedge q) \rightarrow \Box\bot)$ are provable in L. Set $\varphi = \Box(\neg q \wedge r)$ and $\psi = \Box(p \wedge q) \rightarrow \Box\bot$. Then, using the following proof tree in **GEC**:

$$\frac{p \wedge q, \neg q \wedge r \Rightarrow \bot \qquad \bot \Rightarrow p \wedge q \qquad \bot \Rightarrow \neg q \wedge r}{\Box(p \wedge q), \Box(\neg q \wedge r) \Rightarrow \Box\bot} EC$$

we see that the formula $\varphi \rightarrow \psi$ is provable in EC and hence in ECN. Let G be either **GEC** or **GECN** and assume that the interpolant θ exists. Hence, the sequents $\Box(\neg q \wedge r) \Rightarrow \theta$ and $\Box(p \wedge q), \theta \Rightarrow \Box\bot$ are both provable in G. To show that the existence of θ is contradictory, we will first analyse the general form of θ.
First, note that by a simple induction on the structure of the formulas, it is possible to show that any formula A is **G3cp**-equivalent to a CNF-style formula $\bigwedge_{i \in I} \bigvee_{j \in J_i} L_{ij}$, where I and J_i's are (possibly empty) finite sets, $V(L_{ij}) \subseteq V(A)$, and each L_{ij} is either an atomic formula, the negation of an atomic formula, $\Box C$ or $\neg\Box C$, for a formula C. In particular, the formula θ is **G3cp**-equivalent to a CNF-style formula in the form $\bigwedge_{i \in I} \bigvee_{j \in J_i} L_{ij}$. W.l.o.g, assume that for any $i \in I$, it is impossible to have both an atomic formula and its negation in $\{L_{ij}\}_{j \in J_i}$, and that none of sequents $(\Rightarrow L_{ij})$ or $(L_{ij} \Rightarrow)$ are provable in G.

Back to the main argument, as $\varphi \Rightarrow \theta$ is provable in G, we have $\varphi \Rightarrow \bigwedge_{i \in I} \bigvee_{j \in J_i} L_{ij}$ which means that for every $i \in I$, we have $\varphi \Rightarrow \bigvee_{j \in J_i} L_{ij}$. Based on the form of each L_{ij}, we can transform the sequent to a provable sequent of the form $\varphi, P, \Box\Gamma \Rightarrow$

$Q, \Box\Delta$, where P and Q are multisets of atomic formulas and Γ and Δ are multisets of formulas. We claim that Γ is non-empty. Suppose $\Gamma = \varnothing$. Then, we have $\varphi, P \Rightarrow Q, \Box\Delta$. This sequent must have been the conclusion of the rule (EC), because for $G = \mathbf{GEC}$, the other possible case is being an axiom which implies either $\bot \in P$ or the existence of an atomic s in $P \cap Q$. Both contradict the structure of $\bigvee_{j \in J_i} L_{ij}$. For $G = \mathbf{GECN}$, the same holds. Moreover, if the last rule is (NW), then for an element $\delta \in \Delta$, the sequent $(\Rightarrow \delta)$ and hence $(\Rightarrow \Box\delta)$ must be provable in G which contradicts the structure of L_{ij}'s again. Therefore, $T = (\varphi, P \Rightarrow Q, \Box\Delta)$ is the consequence of (EC) and hence, it has the form $(\Sigma, \Box\alpha_1, \cdots, \Box\alpha_n \Rightarrow \Box\beta, \Lambda)$ and the last rule is:

$$\frac{\alpha_1, \cdots, \alpha_n \Rightarrow \beta \qquad \beta \Rightarrow \alpha_1 \qquad \cdots \qquad \beta \Rightarrow \alpha_n}{\Sigma, \Box\alpha_1, \cdots, \Box\alpha_n \Rightarrow \Box\beta, \Lambda} \; EC$$

Now there are two cases to consider, either $\varphi \in \Sigma$ or $\varphi \notin \Sigma$. In the first case, as the formulas outside of Σ are either atomic or boxed, we must have no boxed formula outside of Σ. This is impossible, as the form of the rule (EC) dictates that we must have at least one boxed formula in the antecedent of the conclusion. Hence, $\varphi \notin \Sigma$. As all formulas in T^a (except φ) are atomic, we must have only one boxed formula in T^a, which is φ. Therefore, in the premises of the rule, we have $\neg q \wedge r \Rightarrow \beta$ and $\beta \Rightarrow \neg q \wedge r$. Since $V(\beta) \subseteq V(\theta) \subseteq \{q\}$, then β is r-free. If we once substitute $\bot$ for r and then $\neg q$ for r, as β remains intact, we will have $\beta \Leftrightarrow \bot$ and $\beta \Leftrightarrow \neg q$, which implies the contradictory $\bot \Leftrightarrow \neg q$. Hence, Γ cannot be empty.

So far, we have proved that Γ is non-empty. Suppose that for every $i \in I$, the formula L_{ik_i} has the negative form $L_{ik_i} = \neg\Box D_i$. Now, as $\Box(p \wedge q), \theta \Rightarrow \Box\bot$ or equivalently $\Box(p \wedge q), \bigwedge_{i \in I} \bigvee_{j \in J_i} L_{ij} \Rightarrow \Box\bot$ is provable in G, we have $\Box(p \wedge q), \{\neg\Box D_i\}_{i \in I} \Rightarrow \Box\bot$ is provable in G. Define $\mathcal{D} = \{D_i\}_{i \in I}$. Thus $S = (\Box(p \wedge q) \Rightarrow \Box\mathcal{D}, \Box\bot)$ is provable. As all the formulas are boxed, this must have been the conclusion of the rule (EC). The reason is that G has no weakening rules, and for $G = \mathbf{GEC}$, the only modal rule is (EC) and for $G = \mathbf{GECN}$, the last rule cannot be the rule (NW) as it implies that for one $D \in \mathcal{D}$ the sequent $(\Rightarrow D)$ is provable in G which means that $(\Rightarrow \Box D)$ and hence $(\neg\Box D \Rightarrow)$ is provable. The last contradicts with the structure of L_{ij}'s. This implies that the last inference is of the form:

$$\frac{\alpha_1, \cdots, \alpha_n \Rightarrow \beta \qquad \beta \Rightarrow \alpha_1 \qquad \cdots \qquad \beta \Rightarrow \alpha_n}{\Sigma, \Box\alpha_1, \cdots, \Box\alpha_n \Rightarrow \Box\beta, \Lambda} \; EC$$

Similar as before, there are two cases, either $\beta = \bot$ or $\beta \in \mathcal{D}$. If $\beta = \bot$, in the premises we must have $p \wedge q \Leftrightarrow \bot$ which is impossible. If $\beta \in \mathcal{D}$, it means that in the premises we had $p \wedge q \Leftrightarrow \beta$. Note that as $\beta \in \mathcal{D}$ we have $V(\beta) \subseteq V(\theta) \subseteq \{q\}$. Hence β is p-free. Substituting once $\bot$ and then q for p, leave β intact and hence we get $\bot \Leftrightarrow \beta$ and $q \Leftrightarrow \beta$ which implies $q \Leftrightarrow \bot$, which is impossible.

Theorem 4.1. *Logics* EC *and* ECN *do not enjoy the Craig interpolation property. As a consequence, they do not have uniform or uniform Lyndon interpolation property.*

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