Universal Proof Theory: Semi-analytic Rules and Interpolation

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Abstract

In [7] and [8], Iemhoff introduced a connection between the existence of a terminating sequent calculus of a certain kind and the uniform interpolation property of the super-intuitionistic logic that the calculus captures. In this paper, we will generalize this relationship to also cover the substructural setting on the one hand and a more powerful type of systems called semi-analytic calculi, on the other. To be more precise, we will show that any sufficiently strong substructural logic with a semi-analytic calculus has Craig interpolation property and in case that the calculus is also terminating, it has uniform interpolation. This relationship then leads to some concrete applications. On the positive side, it provides a uniform method to prove the uniform interpolation property for the logics $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, **IPC**, **CPC** and some of their **K** and **KD**-type modal extensions. However, on the negative side the relationship finds its more interesting application to show that many substructural logics including L_n, G_n, BL, R and RM^e, almost all super-intutionistic logics (except at most seven of them) and almost all extensions of **S4** (except thirty seven of them) do not have a semi-analytic calculus. It also shows that the logic **K4** and almost all extensions of the logic **S4** (except six of them) do not have a terminating semi-analytic calculus.

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Contents

1 Introduction

Proof systems have the main role in any proof theoretic investigation, from Gentzen's consistency proof and Kreisel's proof mining program to the characterizations of the admissible rules of the logical systems and their decidability problems. In this respect, proof systems are nothing but some technical tools in the study of their corresponding mathematical theories. They are designed and used based on their expected applications and not their inherent mathematical values. They are just the second rank citizens, far from the independent mathematical objects that they could have been.

Fortunately, in the recent years, alongside this instrumentalist approach, another approach has also been emerged; an approach that is more interested in the general behaviour of the proof systems than their possible technical applications (for instance, see [7], [8] and [3]). We call this emerging approach, the *universal proof theory*;[1] a name we hope to be reminiscent of the technical term *universal algebra* used for the theory that is supposed to investigate the generic behaviour of the algebraic structures. This theory is admittedly a hypothetical theory, but whatever it turns out to be, its agenda may include the following fundamental problems:

(*i*) The *existence problem* to investigate the existence of the different sorts of interesting proof systems such as the terminating systems, the normalizable systems, etc.

(*ii*) The *equivalence problem* to investigate the natural notions of equivalence between proof systems. This can be interpreted as an approach to address the so-called Hilbert's twenty fourth problem of studying the equivalence of different mathematical proofs, rigorously.

(*iii*) And finally, the *characterization problem* to investigate the possible characterizations of proof systems via a given equivalence relation as introduced in (*ii*).

As the first step in this so-called universal proof theory and following the spirit of [7] and [8], we begin with the most basic problem of the kind, the *existence problem*, addressing the existence of the natural sequent style proof

[1]We are grateful to Masoud Memarzadeh for this elegant terminological suggestion.

systems for a given propositional and modal logic. The technique is developing a strong relationship between the existence of some sort of proof systems and some regularity conditions for the logic that it represents. One loose example of such a relationship is the relationship between the existence of a terminating calculus for a logic and its decidability. These relationships are important because they reduce the existence problem partially or completely to the regularity conditions of the logic that are calculus-independent and probably more amenable to our technical tools. Again using our loose example, we know that an undecidable logic cannot have a terminating calculus; a fact that solves the existence problem negatively.

This paper is devoted to one of these kinds of relationships and to explain how, we have to browse the history a little bit, first. The story begins with Pitts' seminal work, [11], in which he introduced a proof theoretic method to prove the uniform interpolation property for the propositional intuitionistic logic. His technique is built on the following two main ideas: First he extended the notion of uniform interpolation from a logic to its sequent calculus in a way that the uniform p-interpolants for a sequent are roughly the best left and right p-free formulas that if we add them to the left or right side of the sequent, they make the sequent provable. This reduces the task of proving uniform interpolation for the logic, to the task of finding these new uniform interpolants for all sequents. For the latter, he assigned two sets of p-free formulas to any sequent using the structure of the formulas occurred in the sequent itself. To define these sets, though, he needed the second crucial tool of the game namely the terminating calculus for **IPC**, introduced in [4] by Dyckhoff. The terminating calculus provides a well-founded order on sequents on which we can define the sets that we have mentioned before, recursively.

Later, as witnessed in [8], Iemhoff recognized that the main point in the first part of Pitts' argument is flexible enough to apply on any rule with a certain general form. This observation then lets her to lift the technique from the intuitionistic logic to any extension of the intuitionistic logic presented with a generic terminating calculus consisting of the usual axioms of the calculus **LJ** and the above-mentioned rules that she calls focused axioms and focused rules, respectively. These are the rules that are very natural to consider and they are roughly the rules with one main formula in their consequence such that the rule respects both the side of this main formula and the occurrence of atoms in it, i.e. if the main formula is occurred in the left-side (right-side) of the consequence, all non-contextual formulas in the premises should also occur in the left-side (right-side) and any occurrence of any atom in these formulas must also occur in the main formula. The usual conjunction and disjunction rules are the prototype examples of these rules while the implication rules are the non-examples since they clearly do not respect the side of the main formula.

As we explained, the investigations in [8] lead to an exciting relationship between the existence of a terminating calculus consisting only of the focused axioms and focused rules for a logic and the uniform interpolation property of the logic. Iemhoff used this relationship first in a positive manner to prove the uniform interpolation for some well-known super-intuitionistic and super-intuitionistic modal logics including **IPC**, **CPC**, **K** and **KD** and their intuitionistic versions. And then she switched to the negative part to show that no extension of the intuitionistic logic can have a terminating calculus consisting of focused axioms and focused rules unless it has the uniform interpolation property. Since uniform interpolation is a rare property for a logic, it excludes almost all logical systems, including all super-intuitionistic logics, except the seven logics with the uniform interpolation property, from having such a terminating calculus.

Now we are ready to explain what we will pursue in this paper. Our approach is a generalization of the approach in [7] and [8], in the following three aspects: First we use a much more general class of rules that we will call semi-analytic rules. These rules can be defined roughly as the focused rules relaxing the side preserving condition. Therefore, they cover a vast variety of rules including focused rules, implication rules, non-context sharing rules in substructural logics and so many others. Secondly, we generalize the focused axioms of [8] to cover more general forms of axioms. And finally, we lower the base logic from the intuitionistic logic to the basic substructural logic $\mathbf{FL_e}$ to extend the applicability of the final result to cover substructural logics as well.

After these generalizations, as in [8], our main result connects the existence of proof systems consisting of semi-analytic rules and focused axioms to a strong version of Craig interpolation property called the feasible interpolation and in the case that the system is also terminating to an even stronger form of uniform interpolation. As it is expected, this connection also has two sorts of applications. First on the positive side, it says that if we manage to develop a terminating calculus consisting of semi-analytic rules and focused axioms, there is a uniform method to establish the uniform interpolation property. The logics with this property include some substrucutral logics like $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$ and their **K** and **KD** modal extensions and intuitionistic and classical logics and some of their modal extensions. (For the classical modal case see [2], for the substructural logics see [1] and for intuitionistic and intutionistic modal logics see [11] and [8].) Moreover, note that there is a possibility that we manage to develop a system of the mentioned form that fails to be terminating. In this case the connection is still useful but only to establish the Craig interpolation. The logics in this category include **K4** and **S4**-type of modal extensions of some substructural logics including the intutionistic and classical linear logics in which the exponentials play the role of the **S4**-type modality.

Despite the possible use of the positive applications of the connection, it is fair to say that developing a uniform method to prove interpolation is not very useful. The reason is the common knowledge that it is genuinely rare for a logic to have the interpolation property. To justify this feeling, note that in the substructural setting, there are a lot of relevant and semilinear logics ([12], [10]) that lack this property and as we have already seen in the super-intutionistic case, there is a well-known result by Maksimova [9] stating that among super-intuitionistic logics, there are only seven specific logics that have Craig or uniform interpolation.

Using this insight, we will turn the relationship between the interpolation and the existence of proof systems to its negative side to propose the main contribution of this paper. We will use the connection to show that logics without Craig interpolation do not have a calculus consisting only of semi-analytic rules and focused axioms and if they have Craig interpolation but fail to have uniform interpolation, the proof system if exists will not be terminating. Given the generality of these rules and axioms, this negative application excludes so many logics from having a reasonable proof system. To name a few concrete examples consider the logics L_n, G_n, BL, R and RM^e in the substructural world, all super-intuitionistic logics except **IPC**, **LC**, **KC**, $\mathbf{Bd_2}$, **Sm**, **GSc** and **CPC** in the super-intuitionistic domain and all extensions of **S4** except at most thirty seven of them in the modal case. In the uniform case, there are also some concrete examples including the logics

K4 and all the extensions of **S4** except at most six of them for which our result shows the non-existence of a terminating calculus consisting only of semi-analytic rules and focused axioms.

2 Preliminaries

In this section we will cover some of the preliminaries needed for the following sections. The definitions are similar to the same concepts in [8] and [10], but they have been changed whenever it is needed.

In the following, we define a translation between two arbitrary languages. The reason for using such a notion is that in the upcoming sections we will consider logics with a fixed but an arbitrary language. This is a generalization which makes our results much stronger since their importance is that they are negative results. Therefore, the broader the range of the logics is, the stronger the results will be.

Definition 2.1. Let us denote $p_1, \ldots, p_n$ by $\bar{p}$, where each p_i is an atomic formula. Let $\mathcal{L}$ and $\mathcal{L}'$ be two languages. By a *translation* $t : \mathcal{L} \to \mathcal{L}'$, we mean an assignment which assigns a formula $\phi_C(\bar{p}) \in \mathcal{L}'$ to any logical connective $C(\bar{p}) \in \mathcal{L}$ such that any p_i has at most one occurrence in $\phi_C(\bar{p})$. It is possible to extend a translation from the basic connectives of the language to all of its formulas in an obvious compositional way. We will denote the translation of a formula ϕ by ϕ^t and the translation of a multiset Γ, by $\Gamma^t = \{\phi^t \mid \phi \in \Gamma\}$.

In this paper, we will work with a fixed but arbitrary language $\mathcal{L}$ that is augmented by a translation $t : \{\wedge, \vee, \to, *, 0, 1\} \cup \mathcal{L} \to \mathcal{L}$ in the single-conclusion cases and by $t : \{\wedge, \vee, \to, *, +, 0, 1\} \cup \mathcal{L} \to \mathcal{L}$ in multi-conclusion cases, that fixes all logical connectives in $\mathcal{L}$. For this reason and w.l.o.g, we will assume that the language already contains the connectives $\{\wedge, \vee, \to, *, 0, 1\}$ in single-conclusion cases and $\{\wedge, \vee, \to, *, +, 0, 1\}$ in multi-conclusion ones. In the case of modal logics, the language $\mathcal{L}$ will be extended to contain the modal operator $\Box$, as well.

Example 2.2. The usual language of classical propositional logic is a valid language in our setting. In this case, there is a *canonical translation* that sends fusion, addition, 1 and 0 to conjunction, disjunction, $\top$ and $\bot$, respectively. In this paper, whenever we pick this language, we assume that we are working with this canonical translation.

Definition 2.3. By a logic L in the language $\mathcal{L}$, we mean a subset of the set of all $\mathcal{L}$-formulas that is closed under arbitrary substitution and the following rules:

- the modus ponens rule, i.e., if $\phi, \phi \to \psi \in \mathsf{L}$, then $\psi \in \mathsf{L}$, and
- the adjunction rule, i.e., if $\phi, \psi \in \mathsf{L}$, then $\phi \wedge \psi \in \mathsf{L}$.

2.1 Sequent Calculi

We denote atomic formulas by small Roman letters, $p, q, \ldots$. Formulas are defined in the usual way from atomic formulas and atomic constants and connectives in the language, and we denote them by small Greek letters $\phi, \psi, \ldots$ or by capital Roman letters $A, B, \ldots$. We denote multisets of formulas by capital Greek letters $\Gamma, \Delta, \ldots$ and we mean the order does not matter but the multiplicity of formulas is important. However, sometimes we use the bar notation for multisets to make everything simpler. For instance, by $\bar{\phi}$, we mean a multiset consisting of formulas $\phi_1, \ldots, \phi_n$. We denote the number of elements (cardinality) of the multiset Γ by $|\Gamma|$. By $\Gamma \cup \Delta$ or Γ, Δ we mean the multiset containing all the formulas ϕ which is in Γ or in Δ. By a *sequent*, we mean an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas in the language. By a *single-conclusion* sequent $\Gamma \Rightarrow \Delta$ we mean that the multiset Δ contains at most one formula, and we call it *multi-conclusion* otherwise. In the single-conclusion cases a sequent $\Gamma \Rightarrow \Delta$ is interpreted as $\mathop{\ast}\Gamma \to \Delta$, and if $\Delta = \varnothing$ as $\mathop{\ast}\Gamma \to 0$, and in the multi-conclusion cases it is interpreted as $\mathop{\ast}\Gamma \to \mathop{+}\Delta$, where by $\mathop{\ast}\Gamma$ we mean the formula $\gamma_1 * \gamma_2 * \ldots * \gamma_n$, where each $\gamma_i \in \Gamma$; the formula $\mathop{+}\Delta$ is defined similarly.

For a sequent $S = (\Gamma \Rightarrow \Delta)$, by S^a we mean the antecedent of the sequent, which is Γ, and by S^s we mean the succedent of the sequent, which is Δ. The *multiplication* of two sequents S and T is defined as $S \cdot T = (S^a \cup T^a \Rightarrow S^s \cup T^s)$.
Meta-language, $\widehat{\mathcal{L}}$, is the language in which we define the sequent calculi. It consists of infinitely many formula variables $\hat{\phi}, \hat{\psi}, \ldots$, the logical connectives $\wedge, \vee, \to, *$ (and $+$ in the multi-conclusion cases and $\Box$ in modal cases), and

constants $0, 1, \bot, \top$. *Meta-formulas* are defined as usual: all formula variables, atomic formulas and constants are meta-formulas and if ϕ and ψ are meta-formulas, so is $\phi \circ \psi$ for $\circ \in \{\wedge, \vee, \rightarrow, *\}$ (and $\phi + \psi$ in multi-conclusion cases and $\Box\phi$ in modal cases). We have also an infinite number of *meta-multiset variables*, also called *contexts*, which are denoted by $\widehat{\Gamma}, \widehat{\Delta}, \ldots$. A *meta-multiset* is a multiset containg meta-formulas and meta-multiset variables. A *meta-sequent* is an expression of the form $\widehat{S} = X \Rightarrow Y$ such that X and Y contain finite number of meta-formulas and meta-multisets. The *set of variables* of a meta-formula ϕ, $V(\phi)$, is defined inductively. For any constant c in the language, $V(c)$ is defined as the empty set. For an atomic formula p and for a formula variable $\hat{\phi}$, define $V(p) = p$ and $V(\hat{\phi}) = \hat{\phi}$. For a logical connective $\circ \in \{\wedge, \vee, \rightarrow, *, +, \backslash, /\}$ define $V(\phi \circ \psi)$ as $V(\phi) \cup V(\psi)$. Moreover, $V(\Box\phi) = V(\phi)$, and $V(\Gamma) = \{V(\phi) \mid \phi \in \Gamma\}$ for a meta-multiset Γ. A meta-formula ϕ is called *p-free*, for an atomic formula or meta-formula variable p, when $p \notin V(\phi)$.

A *substitution* σ is a map from the union of meta-multisets and meta-formulas in $\widehat{\mathcal{L}}$ to the union of multisets and formulas in $\mathcal{L}$ that works as follows: constants are mapped to themselves, meta-formulas to formulas, meta-multisets to multisets, and σ commutes with the logical connectives and the modal operator. Therefore, $\sigma(\hat{\phi})$ will be a formula in $\mathcal{L}$, $\sigma(\widehat{\Gamma})$ will be the multiset of formulas $\sigma(\hat{\gamma})$, where $\hat{\gamma} \in \widehat{\Gamma}$, and $\sigma(\widehat{S} = X \Rightarrow Y)$ will be $\sigma(X) \Rightarrow \sigma(Y)$.

A *rule* is an expression of the form

$$\frac{\widehat{S}_1, \cdots, \widehat{S}_n}{\widehat{S}}$$

where $\widehat{S}, \widehat{S}_1, \ldots, \widehat{S}_n$ are meta-sequents. Meta-sequents above the line are called *premisses* and the one below the line, the *conclusion*. In the case the rule has no premises, it is called an *axiom*. It is called a *left* (*right*) rule if $\widehat{S}^a$ ($\widehat{S}^s$) contains a meta-formula. A rule is either a right rule or a left one. An *instance* of a rule is obtained by using the substitution map on the rule as follows

$$\frac{\sigma(\widehat{S}_1), \cdots, \sigma(\widehat{S}_n)}{\sigma(\widehat{S})}$$

Note that if there is a side condition on the rule, such as the meta-formulas must everywhere be atoms, this condition works as a restriction on the substitution σ. A rule is *backward applicable* to a sequent S, when there is at

least one instance of the rule where S is the conclusion.
By a *sequent calculus* $\mathbf{G}$, we mean a set of rules. We will use bold-face capital Roman letters to denote sequent calculi. A sequent S is *derivable* from a set of sequents Γ in $\mathbf{G}$, denoted by $\Gamma \vdash_G S$, if there exists a finite tree with sequents as labels of the nodes such that the label of the root is S, labels of the leaves are axioms of $\mathbf{G}$ or members of Γ, and in each node the set of the labels of the children of the node together with the label of the node itself, constitute an instance of a rule in $\mathbf{G}$. This finite tree is called the *proof* of S in $\mathbf{G}$ which is sometimes called a tree-like proof to emphasize its tree-like form. If $\Gamma = \varnothing$ then we denote it by $\mathbf{G} \vdash S$ and we say S is derivable in $\mathbf{G}$. We will use the same notation for a sequent calculus and its logic, i.e., the set of provable formulas in it, i.e., $\{\phi \mid G \vdash (\Rightarrow \phi)\}$.

As it is usually a convention in proof theory papers, from now on we will not mention "meta" in the meta-language and so on and we will omit the $\hat{}$ notation. It will be always clear from the context which form we are working with. Therefore, for instance by a meta-sequent $\Gamma, \bar{\phi} \Rightarrow \psi$, we mean Γ is a meta-multiset, $\bar{\phi}$ is a possibly empty multiset of meta-formulas and ψ is a meta-formula.

Let us recall some important systems that we will use throughout the paper. Consider the following set of rules:

Identity:

$$\frac{}{\phi \Rightarrow \phi}$$

Context-free Axioms:

$$\frac{}{\Rightarrow 1} \qquad \frac{}{0 \Rightarrow}$$

Rules for 0 and 1:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, 1 \Rightarrow \Delta}\,(1w) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow 0, \Delta}\,(0w)$$

Conjunction Rules:

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\,(L\wedge) \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\,(L\wedge) \qquad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta}\,(R\wedge)$$

Disjunction Rules:

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta}\,(L\vee) \quad \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\,(R\vee) \quad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\,(R\vee)$$

Fusion Rules:

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi * \psi \Rightarrow \Delta}\,(L*) \quad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma \Rightarrow \psi, \Lambda}{\Gamma, \Sigma \Rightarrow \phi * \psi, \Delta, \Lambda}\,(R*)$$

Implication Rules:

$$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi \rightarrow \psi \Rightarrow \Delta, \Lambda}\,(L \rightarrow) \quad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta}\,(R \rightarrow)$$

The system consisting of the single-conclusion version of all of these rules is $\mathbf{FL_e}^-$. If we also add the single-conclusion version of the following axioms, we reach a system which we denote by $\mathbf{FL_e^b}$.

Contextual Axioms:

$$\frac{}{\Gamma \Rightarrow \top, \Delta} \quad \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

In the standard definition of $\mathbf{FL_e}$ the language does not contain the constants $\bot$ and $\top$ and therefore their axioms are not present in the sequent calculus, as well. However, since the presence of $\bot$ and $\top$ is essential in our discussions in the future sections, we allow them in the language and their axioms in the sequent calculus.

In the multi-conclusion case define $\mathbf{CFL_e}^-$ and $\mathbf{CFL_e}$ with the same rules as $\mathbf{FL_e}^-$ and $\mathbf{FL_e}$, this time in their full multi-conclusion version and add $+$ to the language and the following rules to the systems:

Rules for $+$:

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi + \psi \Rightarrow \Delta, \Lambda}\,(L+) \quad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi + \psi, \Delta}\,(R+)$$

The system **MALL** is defined as $\mathbf{CFL_e}$ minus the implication rules. Moreover, if we consider the following rules:

$$\frac{!\Gamma \Rightarrow \phi}{!\Gamma \Rightarrow !\phi}\,\dagger \quad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta} \quad \frac{\Gamma, !\phi, !\phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

we can define **ILL** as $\mathbf{FL_e}$ plus the single-conclusion version of the above rules and **CLL** as $\mathbf{CFL_e}$ plus the above rules, themselves. In both cases, the rule † is single-conclusion. The set of provable formulas in any of the sequent calculi defined above, i.e., their corresponding logics, are denoted by $\mathsf{FL_e^-}, \mathsf{CFL_e^-}, \mathsf{FL_e}, \mathsf{CFL_e}, \mathsf{MALL}, \mathsf{ILL}$, and CLL.
We will use later the structural rules given below:

Weakening rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\,(Lw) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}\,(Rw)$$

Note that in the single-conclusion cases, in the rule (Rw), Δ must be empty.

Contraction rules:

$$\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\,(Lc) \quad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \phi, \Delta}\,(Rc)$$

The rule (Rc) is only allowed in multi-conclusion systems.

If we consider the sequent calculus $\mathbf{FL_e}$ and add the weakening rules (contraction rules), the resulting system is called $\mathbf{FL_{ew}}$ ($\mathbf{FL_{ec}}$). Their corresponding logics are denoted by $\mathsf{FL_{ew}}$ and $\mathsf{FL_{ec}}$. In a similar manner, we define $\mathbf{CFL_{ew}}$ and $\mathbf{CFL_{ec}}$, and their corresponding logics $\mathsf{CFL_{ew}}$ and $\mathsf{CFL_{ec}}$. Finally, adding all the structural rules to $\mathbf{FL_e}$, we obtain the system $\mathbf{FL_{ewc}}$ in which the connectives $*$ and $\wedge$ become equivalent, i.e., $\phi * \psi \Leftrightarrow \phi \wedge \psi$ will become provable in the system. Moreover, $\bot$ and 0, and $\top$ and 1 will become equivalent in $\mathbf{FL_{ewc}}$. Furthermore, in the system $\mathbf{CFL_{ewc}}$, we can also prove that $+$ and $\vee$ are equivalent. Hence, it is possible to define $\mathbf{FL_{ewc}}$ ($\mathbf{CFL_{ewc}}$) even on the restricted language $\{\wedge, \vee, \top, \bot, \rightarrow\}$. This system is nothing but the usual sequent calculus **LJ** (**LK**) for the intuitionistic (classical) logic IPC(CPC). All the sequent calculi presented here enjoy the cut-elimination property [].

We will also use the following rules in the future sections:

Context-sharing left implication:

$$\frac{\Gamma \Rightarrow \phi \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \to \psi \Rightarrow \Delta}$$

Left weakening rule for boxed formulas:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Box\phi \Rightarrow \Delta}$$

Modal rules:

$$\frac{\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\ \mathsf{K} \qquad \frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\ \mathsf{D}$$

$$\frac{\Box\Gamma, \Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\ \mathsf{4} \qquad \frac{\Box\Gamma, \Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\ \mathsf{4D}$$

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Box\phi \Rightarrow \Delta}\ L\mathsf{S4} \qquad \frac{\Box\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\ R\mathsf{S4}$$

We use the convention that $\Box\varnothing = \varnothing$. We say a sequent calculus $\mathbf{G}$ satisfies *the modal admissibility conditions* if the following conditions hold in $\mathbf{G}$:

- if the rule (D) is present in $\mathbf{G}$, the rule (K) is admissible in it;
- if the rule ($\mathsf{4D}$) is present in $\mathbf{G}$, the rule ($\mathsf{4}$) is admissible in it, as well, and
- if the rule ($R\mathsf{S4}$) is present in $\mathbf{G}$, the rule ($L\mathsf{S4}$) is admissible in it.

Adding these rules to the above sequent calculi does not affect the cut-elimination property [].

Finally, note that Γ and Δ are multiset variables everywhere, therefore the exchange rule is built in and hence admissible in our system. Moreover, note that the calculi defined in this section are written in the given language which can be any extension of the language of the system itself. For instance, $\mathbf{FL_e}$ is the calculus with the mentioned rules on our fixed language that can have more connectives than $\{\wedge, \vee, *, \to, \top, \bot, 1, 0\}$.
By a subsequent of a sequent $\Gamma \Rightarrow \Delta$ we mean a sequent $\Gamma' \Rightarrow \Delta'$. We call it proper if either $\Gamma' \subsetneq \Gamma$ or $\Delta' \subsetneq \Delta$.

Definition 2.4. A sequent calculus G is *terminating with respect to* $<$, where $<$ is a well-founded order on the sequents, G is finite and there are at most finitely many instances of the rules in G with the conclusion S. Moreover, the order is defined in a way that the order of the following are less than the order of S:

- the premises of all instance of a rule whose conclusion is S;
- proper subsequents of S, and
- any sequent S' of the form $(\Gamma, \Pi \Rightarrow \Delta, \Lambda)$, where S is of the form $(\Gamma, \Box\Pi \Rightarrow \Delta, \Box\Lambda)$. Note that $\Pi \cup \Lambda$ must be non-empty.

Definition 2.5. Let $\mathbf{G}$ be a sequent calculus and L be a logic such that they have the same language. We say $\mathbf{G}$ is a *sequent calculus for* L when

$$\mathbf{G} \vdash \Gamma \Rightarrow \Delta \quad \text{if and only if} \quad \mathsf{L} \vdash (\circledast \Gamma \rightarrow \textstyle\bigplus \Delta).$$

Note that if the calculus is single-conclusion, by $\bigplus \Delta$, we mean Δ if Δ is a singleton, and 0 if Δ is empty. Therefore, in this case we do not need the $+$ operator. As a result of Definition 2.5, if $\mathbf{G}$ is a sequent calculus for L we have

$$\mathbf{G} \vdash \phi \Rightarrow \psi \text{ iff } \mathsf{L} \vdash \phi \rightarrow \psi.$$

2.2 Logical Systems

In this subsection we will recall the Craig interpolation property, the uniform interpolation property and also some useful substructural logics that we will need in the rest of the paper.

Definition 2.6. We say that a logic L has *Craig interpolation* property if for any formulas ϕ and ψ if $\mathsf{L} \vdash \phi \rightarrow \psi$, then there exists a formula θ such that $\mathsf{L} \vdash \phi \rightarrow \theta$ and $\mathsf{L} \vdash \theta \rightarrow \psi$ and $V(\theta) \subseteq V(\phi) \cap V(\psi)$.

Definition 2.7. We say a logic L has the *uniform interpolation* property if for any formulas ϕ and any atomic formula p, there are two p-free formulas, the p-pre-interpolant, $\forall p\phi$ and the p-post-interpolant $\exists p\phi$, such that $V(\exists p\phi) \subseteq V(\phi)$ and $V(\forall p\phi) \subseteq V(\phi)$ and

(*i*) $\mathsf{L} \vdash \forall p\phi \rightarrow \phi$,

(*ii*) For any p-free formula ψ if $\mathsf{L} \vdash \psi \rightarrow \phi$ then $\mathsf{L} \vdash \psi \rightarrow \forall p\phi$,

(*iii*) $\mathsf{L} \vdash \phi \rightarrow \exists p\phi$, and

(*iv*) For any p-free formula ψ if $\mathsf{L} \vdash \phi \rightarrow \psi$ then $\mathsf{L} \vdash \exists p\phi \rightarrow \psi$.

To recall some of the well known substructural logics and following [10], we have to introduce the semantical framework, first.

Definition 2.8. By a *pointed commutative residuated lattice* we mean an algebraic structure $\mathbf{A} = \langle A, \wedge, \vee, *, \to, 0, 1\rangle$ where $\wedge, \vee, *, \to$ are binary operations, and $0, 1$ are constants such that $\langle A, \wedge, \vee\rangle$ is a lattice with partial order $\leqslant$ and $\langle A, *, 1\rangle$ is a commutative monoid. We define for all $x, y, z \in A$, $x * y \leqslant z$ if and only if $x \leqslant y \to z$. For a single pointed commutative residuated lattice $\mathbf{A}$ and a class of pointed commutative residuated lattices $\mathbf{K}$, denote $\mathcal{V}(\mathbf{A})$ and $\mathcal{V}(\mathbf{K})$ as the varieties generated by $\mathbf{A}$ and $\mathbf{K}$, respectively.

In the following we will borrow the definitions of some logics from [10]. First, we need the following equational conditions for pointed commutative residuated lattices.

- (prl) prelinearity : $1 \leqslant (x \to y) \vee (y \to x)$
- (dis) distributivity : $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- (inv) involutivity : $\neg\neg x = x$
- (int) integrality : $x \leqslant 1$
- (bd) boundedness : $0 \leqslant x$
- (id) idempotence : $x = x * x$
- (fp) fixed point negation : $0 = 1$
- (div) divisibility : $x * (x \to y) = y * (y \to x)$
- (can) cancellation : $x \to (x * y) = y$
- (rcan) restricted cancellation : $1 = \neg x \vee ((x \to (x * y)) \to y)$
- (nc) non-contradiction : $x \wedge \neg x \leqslant 0$

In the following, we have the definitions of some logics that we are interested in. Note that in all of them, both of the axioms (*prl*) and (*dis*) are present, and hence we just mention the other axioms.

- (UL^-) unbounded uninorm logic

- (IUL^-) unbounded involutive uninorm logic : (inv)
- (MTL) monoidal t-norm logic : $(int), (bd)$
- (SMTL) strict monoidal t-norm logic : $(int), (bd), (nc)$
- (IMTL) involutive monoidal t-norm logic : $(int), (bd), (inv)$
- (BL) basic fuzzy logic : $(int), (bd), (div)$
- (G) Gödel logic : $(int), (bd), (id)$
- (L) Lukasiewicz logic : $(int), (bd), (div), (inv)$
- (P) product logic : $(int), (bd), (div), (rcan)$
- (CHL) cancellative hoop logic : $(int), (fp), (div), (can)$
- (UML^-) unbounded uninorm mingle logic : (id)
- ($\mathsf{RM^e}$) R-mingle with unit : $(id), (inv)$
- (IUML^-) unbounded involutive uninorm mingle logic : $(id), (inv), (fp)$
- (A) abelian logic : $(inv), (fp), (can)$

Furthermore, we will define the following important logics, as well.
For $n > 1$ define

$$\mathsf{L}_n = \{0, \tfrac{1}{n-1}, \cdots, \tfrac{n-2}{n-1}, 1\} \qquad , \qquad L_\infty = [0, 1]$$

and the pointed commutative residuated lattices (again for $n > 1$)

$$\mathbf{L_n} = \langle L_n, min, max, *_{\mathsf{L}}, \rightarrow_{\mathsf{L}}, 1, 0\rangle$$

and

$$\mathbf{G_n} = \langle L_n, min, max, min, \rightarrow_G, 1, 0\rangle$$

where $x *_{\mathsf{L}} y = max(0, x + y - 1)$, $x \rightarrow_{\mathsf{L}} y = min(1, 1 - x + y)$, and $x \rightarrow_G y$ is y if $x > y$, otherwise 1. Then, for $n > 1$, L_n and G_n are the logics with equivalent algebraic semantics $\mathcal{V}(\mathbf{L_n})$ and $\mathcal{V}(\mathbf{G_n})$, respectively. The logics G_∞ and $\mathbf{H}_\infty$ are the Gödel logic and Lukasiewicz logic, as defined before.

R is the logic of a variety consisting of all distributive pointed commutative residuated lattices with the condition that $x * x \leqslant x$ for all x.

Now consider the following binary functions on the set of integers $\mathbb{Z}$, where $\wedge$ and $\vee$ are min and max, respectively, and $|x|$ is the absolute value of x:

$$x * y = \begin{cases} x \wedge y & \text{if } |x| = |y| \\ y & \text{if } |x| < |y| \\ x & \text{if } |y| < |x| \end{cases} \qquad x \to y = \begin{cases} -(x) \vee y & \text{if } x \leqslant y \\ -(x) \wedge y & \text{otherwise} \end{cases}$$

And finally define the following algebras:

$$\mathbf{S_{2m}} = \langle \{-m, -m+1, \cdots, -1, 1, \cdots, m-1, m\}, \wedge, \vee, *, \to, 1, -1 \rangle \ \ (m \geqslant 1)$$

$$\mathbf{S_{2m+1}} = \langle \{-m, -m+1, \cdots, -1, 0, 1, \cdots, m-1, m\}, \wedge, \vee, *, \to, 0, 0 \rangle \ \ (m \geqslant 0)$$

and define RM_n^e as the logic of $\mathcal{V}(\mathbf{S_n})$.

3 Semi-analytic Rules

In this section we will introduce a class of rules which we will investigate in the rest of the paper. We will only consider rules with exactly one meta-formula ϕ in the conclusion, which is different from contexts (or multiset variables) Γ_i, Π_j or Δ_i.
By the notation $\langle\langle S_{ir} \rangle_r \rangle_i$, where S_{ir}'s are meta-sequents, we mean first considering the meta-sequents S_{ir} ranging over r and then ranging over i. Moreover, for the sake of simplicity, we omit the domain of indices, while we always mean that $1 \leqslant i \leqslant n$ and $1 \leqslant r \leqslant m_i$. Note that m_i depends on the index i. For instance, $\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i$ is short for the following sequence of meta-sequents where $1 \leqslant i \leqslant n$, and $1 \leqslant r \leqslant m_i$:

$$\Gamma_1, \bar{\phi}_{11} \Rightarrow \bar{\psi}_{11}, \Delta_1, \cdots, \Gamma_1, \bar{\phi}_{1m_1} \Rightarrow \bar{\psi}_{1m_1}, \Delta_1,$$

$$\Gamma_2, \bar{\phi}_{21} \Rightarrow \bar{\psi}_{21}, \Delta_2, \cdots, \Gamma_2, \bar{\phi}_{2m_2} \Rightarrow \bar{\psi}_{2m_2}, \Delta_2,$$

$$\vdots$$

$$\Gamma_n, \bar{\phi}_{n1} \Rightarrow \bar{\psi}_{n1}, \Delta_n, \cdots, \Gamma_n, \bar{\phi}_{nm_n} \Rightarrow \bar{\psi}_{nm_n}, \Delta_n.$$

where each $\bar{\phi}_{ir}$ is a multiset of meta-formulas $\phi_{ir}^1, \ldots, \phi_{ir}^{k_{ir}}$ or the empty sequence and $\bar{\psi}_{ir}$ is a multiset of meta-formulas $\psi_{ir}^1, \ldots, \psi_{ir}^{k'_{ir}}$ or the empty sequence. The reason for such a complicated notation is that we want to be able to talk about the rules in their most general form. Therefore, The premises in a rule may be made of meta-sequents with the same contexts and/or meta-sequents with different contexts. At a closer look, in the ith horizontal line in the definition above, there are m_i sequents with the same contexts Γ_i and Δ_i and possibly different sequences of meta-formulas $\bar{\phi}_{im_i}$ and $\bar{\psi}_{im_i}$, while in vertical lines we also allow the contexts to change. The sequences $\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i$ and $\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j$ are defined similarly. In the former, there are no sequences of meta-formulas in the succedents of the sequents and in the latter, there are no contexts in the succedents of sequents.
Throughout this paper, we will mostly work with sequences in the form $\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j$, $\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i$, or $\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i$ where $\bar{\psi}_{js}, \bar{\theta}_{js}, \bar{\phi}_{ir}$ and $\bar{\psi}_{ir}$ are either the empty sequence or a multiset of meta-formulas and Π_j, Γ_i, and Δ_i are pairwise disjoint sets of multiset variables. In each sequent, Γ_i and Π_j are called the left context while Δ_i is called the right context.

Definition 3.1. A rule is called *occurrence preserving* if any formula variable appeared in any of the premises also appears in the conclusion.

For instance, for the following rule

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

the occurrence preserving condition is

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi).$$

Note that the occurrence preserving condition is defined on the form of the rule and not on an instance of a rule. Therefore, when we say a variable is occurred in the premises we mean in $\bar{\psi}_{js}$, $\bar{\theta}_{js}$ or $\bar{\phi}_{ir}$ and when it appears in the conclusion, it must appear in ϕ.

In the following we will define a class of occurrence preserving rules that we will call *semi-analytic*, since the occurrence preserving condition is the weaker version of the analycity property in the analytic rules, which demands the formulas in the premises to be subformulas of the formulas in the consequence. Based on a rule being single-conclusion, multi-conclusion, context-sharing or a modal rule, the notion of being semi-analytic is defined as follows.

Definition 3.2. Let Γ_i, Π_j and Δ_i be pairwise distinct multiset variables, $\bar{\psi}_{js}$, $\bar{\phi}_{ir}$ and $\bar{\theta}_{js}$ be multisets of meta-formulas and ϕ be a meta-formula where $i \leqslant n$ and $j \leqslant m$. In the left single-conclusion semi-analytic rule, $|\Delta_i| \leqslant 1$ and $\bar{\theta}_{js}$ is either one meta-formula or empty, for every i, j, and s. Also, in the right single-conclusion semi-analytic rule, $\bar{\psi_{ir}}$ is either one meta-formula or empty for each i and r. A rule is called semi-analytic if it is occurrence preserving and has one of the following forms.

- *left single-conclusion semi-analytic*:

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

- *right single-conclusion semi-analytic*:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi}$$

- *context-sharing semi-analytic*:

$$\frac{\langle\langle \Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is} \rangle_s \rangle_i \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

- *left multi-conclusion semi-analytic*:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

- *right multi-conclusion semi-analytic*:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi, \Delta_1, \cdots, \Delta_n}$$

Here are some remarks. First, note that in the left single-conclusion semi-analytic rule since the number of elements of the succedent of the conclusion of the rule must be at most 1, it means that at most one of Δ_i's can be non-empty. Secondly, whenever it is clear from the context, we will omit the phrase "multi-conclusion".

Example 3.3. A generic example of a left semi-analytic rule is the following:

$$\frac{\Gamma, \phi_1, \phi_2 \Rightarrow \psi \qquad \Gamma, \theta \Rightarrow \eta \qquad \Pi, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \Pi, \alpha \Rightarrow \Delta}$$

where

$$V(\phi_1, \phi_2, \psi, \theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha).$$

Note that the premises on the left and in the middle of the example have the same context Γ in the antecedent and have no context in the succedents. Therefore, there should be only one copy of Γ in the antecedent of the conclusion. A generic example of a context-sharing left semi-analytic rule is:

$$\frac{\Gamma, \theta \Rightarrow \eta \qquad \Gamma, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \Delta}$$

where

$$V(\theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha)$$

Moreover, for a generic example of a right semi-analytic rule we can have

$$\frac{\Gamma, \phi \Rightarrow \psi \qquad \Gamma, \theta_1, \theta_2 \Rightarrow \eta \qquad \Pi, \mu_1, \mu_2, \Rightarrow \nu}{\Gamma, \Pi \Rightarrow \alpha}$$

where

$$V(\phi, \psi, \theta_1, \theta_2, \eta, \mu_1, \mu_2, \nu) \subseteq V(\alpha)$$

Here are some remarks. First note that in any left single-conclusion semi-analytic rule there are two types of premises. In the first type, the succedent of the meta-sequent is empty or includes only one meta-formula and in the second type the succedent of the meta-sequent has only one multi-set variable. This is a crucial point to consider. Any left semi-analytic rule allows any kinds of combination of sharing/combining contexts in any type. However, between two types, we can only combine the contexts in the antecedent. The case in which we can share the contexts of the antecedents of meta-sequents of the two types is called context-sharing semi-analytic rule. This should explain why our second example is called context-sharing left semi-analytic while the first one is not. The reason is the fact that the two types share the same context of the antecedent in the second rule while in the first one this situation happens in just one type. The second important point about the semi-analytic rules is the presence of contexts. This is very crucial for almost all the arguments in this paper, that any sequent present in a semi-analytic rule must have multiset variables as left contexts and in the case of left rules, at least one multiset variable for the right hand-side must be present.

Example 3.4. Now for more concrete examples, note that all the usual conjunction, disjunction and implication rules in **LJ** are semi-analytic. The same also holds for all the rules in substructural logic $\mathbf{FL_e}$, the weakening and the contraction rules, the modal rule ($L\mathsf{S4}$), and some of the well-known restricted versions of them including the following rules for exponentials in linear logic:

$$\frac{\Gamma, !\phi, !\phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

For a context-sharing semi-analytic rule, consider the following rule in Dyckhoff's calculus for **IPC** (see [4]):

$$\frac{\Gamma, \psi \to \gamma \Rightarrow \phi \to \psi \qquad \Gamma, \gamma \Rightarrow \Delta}{\Gamma, (\phi \to \psi) \to \gamma \Rightarrow \Delta}$$

Example 3.5. For a concrete non-example consider the cut rule; it is not semi-analytic because it does not preserve the variable occurrence condition. Moreover, the following rule in the calculus of **KC**:

$$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}$$

in which Δ should consist of negation formulas is not a multi-conclusion semi-analytic rule, simply because the context is not free for all possible substitutions. The rule of thumb is that any rule in which we have *side conditions on the contexts* is not semi-analytic.

Definition 3.6. A meta-sequent is called a *focused axiom*, if it has one of the following forms:

(1) Identity axiom: $(\phi \Rightarrow \phi)$

(2) Context-free right axiom: $(\Rightarrow \bar{\alpha})$

(3) Context-free left axiom: $(\bar{\beta} \Rightarrow)$

(4) Contextual left axiom: $(\Gamma, \bar{\phi} \Rightarrow \Delta)$

(5) Contextual right axiom: $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where Γ and Δ are multiset variables and $\bar{\alpha}, \bar{\beta}, \bar{\phi}$ are multisets of meta-formulas and ϕ is a meta-formula. Moreover, in (2) the variables in any pair of elements in $\bar{\alpha}$ are equal, in other words $V(\mu) = V(\nu)$, for any $\mu, \nu \in \bar{\alpha}$. The same condition also holds for any pair of elements in $\bar{\beta}$ in (3) or in $\bar{\phi}$ in (4) and (5). A sequent is called a *context-free focused axiom* if it has one of the forms (1), (2) or (3).

Example 3.7. It is easy to see that the axioms given in the preliminaries are examples of focused axioms. Here are some more examples:

$$\neg 1 \Rightarrow \quad , \quad \Rightarrow \neg 0$$

$$\phi, \neg \phi \Rightarrow \quad , \quad \Rightarrow \phi, \neg \phi$$

$$\Gamma, \neg \top \Rightarrow \Delta \quad , \quad \Gamma \Rightarrow \Delta, \neg \bot$$

where the first four are context-free while the last two are contextual. As a non-example consider $p, \neg p, q \Rightarrow$, where p and q are distinct atomic formulas. It is not a focused axiom since the set of variables of p and q (or $\neg p$ and q) are not equal.

4 Craig Interpolation

In this section we will investigate the relationship between semi-analytic rules and the Craig interpolation property. Apart from its application to prove interpolation for different logics, it will be used to show that several substructural and super-intuitionistic logics cannot have a calculus consisting only of semi-analytic rules and focused axioms. To investigate the relationship between sequent calculi consisting of semi-analytic rules and focused axioms, and the Craig interpolation property we need a notion of comparative interpolation.

Definition 4.1. Let L and L' be two logics such that $\mathcal{L}_L \subseteq \mathcal{L}_{L'}$. We say L' is an *extension* of L (or L' extends L) if $\mathsf{L} \vdash A$ implies $\mathsf{L}' \vdash A$.

Definition 4.2. Let $\mathbf{G}$ and $\mathbf{H}$ be two sequent calculi such that $\mathcal{L}_G \subseteq \mathcal{L}_H$. We say $\mathbf{H}$ is an *extension* of $\mathbf{G}$ if all the rules of $\mathbf{G}$ are admissible in $\mathbf{H}$, i.e., for any instance of a rule of $\mathbf{G}$, if the premises are provable in $\mathbf{H}$ then so is its consequence. Moreover, $\mathbf{H}$ is called an *axiomatic extension* of $\mathbf{G}$, if the provable sequents of $\mathbf{G}$ are considered as axioms of $\mathbf{H}$, to which $\mathbf{H}$ may add some rules.

Definition 4.3. Let $\mathbf{G}$ and $\mathbf{H}$ be two sequent calculi such that $\mathbf{H}$ is an axiomatic extension of $\mathbf{G}$. Let π be a proof of a sequent in $\mathbf{H}$. By the $\mathbf{H}$-*length* of π we mean counting just the new rules that $\mathbf{H}$ adds to the provable sequents in $\mathbf{G}$ that $\mathbf{H}$ considers as axioms.

Theorem 4.4. *Let* L *be a logic and* $\mathbf{G}$ *a single-conclusion (multi-conclusion) sequent calculus for* L. *Then for any logic* $\mathsf{M} \in \{\mathbf{FL_e}^-, \mathbf{FL_e}, \mathbf{IPC}\}$ *(*$\mathsf{M} \in \{\mathbf{CFL_e}^-, \mathbf{CFL_e}\}$*), if we denote the calculus of* M*, defined previously in this section, by* GM*, we have:*

(i) If L *extends* $\mathbf{FL_e}^-$ *(*$\mathbf{CFL_e}^-$*), then the cut rule is admissible in* $\mathbf{G}$*.*

(ii) If L *extends* M*, then* $\mathbf{G}$ *extends the calculus* GM*.*

Proof. First, observe that for any formulas ϕ and ψ, if $\mathsf{L} \vdash \phi$ and $\mathsf{L} \vdash \psi$ then we have $\mathsf{L} \vdash \phi * \psi$. The reason is that L extends $\mathbf{FL_e}^-$ and $\mathbf{FL_e}^- \vdash \phi \to (\psi \to \phi * \psi)$. Therefore, $\mathsf{L} \vdash \phi \to (\psi \to \phi * \psi)$. Since L is closed under modus ponens, if $\mathsf{L} \vdash \phi$ and $\mathsf{L} \vdash \psi$ then $\mathsf{L} \vdash \phi * \psi$.

Now let us prove (i). For the single-conclusion case, set $\bigplus \Delta$ as ϕ when $\Delta = \phi$ and $\bigplus \Delta = 0$, when Δ is empty. Assume that $\mathbf{G} \vdash \Gamma \Rightarrow A, \Delta$ and $\mathbf{G} \vdash \Gamma', A \Rightarrow \Delta'$. Hence $\mathsf{L} \vdash \circledast \Gamma \to A + (\bigplus \Delta)$ and $\mathsf{L} \vdash (\circledast \Gamma') * A \to (\bigplus \Delta')$ by the soundness of $\mathbf{G}$. Therefore, by the previous observation we have

$$\mathsf{L} \vdash [\circledast \Gamma \to A + (\bigplus \Delta)] * [(\circledast \Gamma') * A \to (\bigplus \Delta')]$$

Since L extends $\mathbf{FL_e}^-$ ($\mathbf{CFL_e}^-$) and in this logic the previous formula implies the formula

$$[(\circledast \Gamma) * (\circledast \Gamma') \to (\bigplus \Delta) + (\bigplus \Delta')]$$

By modus ponens in L the last formula is also provable in L which implies $\mathbf{G} \vdash \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ again by the completeness of $\mathbf{G}$.

For (ii), let R be an instance of a rule in the system M and let $S_1, \cdots$, S_n and S_0 be the premises and the consequence of R, respectively. Define $F(\Gamma \Rightarrow \Delta) = [\ast\,\Gamma \to +\,\Delta]$. Then there are three cases to consider:

1. R is an instance of an axiom. Then M proves $F(S_0)$. Since L extends M,we have $\mathsf{L} \vdash F(S_0)$ which implies $\mathbf{G} \vdash S_0$.

2. R is an instance of the conjunction, the disjunction or the structural rules (in this case $\mathsf{M} = \mathbf{IPC}$). Then it is easy to see that the formula

$$\bigwedge_{i=1}^{n} F(S_i) \to F(S_0)$$

is provable in M and hence in L. Now, if $\mathbf{G} \vdash S_i$ for all $1 \leqslant i \leqslant n$, we have $\mathsf{L} \vdash F(S_i)$ which implies $\mathsf{L} \vdash \bigwedge_{i=1}^{n} F(S_i)$ by the adjunction rule. Since L is closed under modus ponens, $\mathsf{L} \vdash F(S_0)$ which implies $\mathbf{G} \vdash S_0$ by the completeness of $\mathbf{G}$.

3. R is an instance of the rules for 0 and 1, the fusion, the addition or the implication rule. Then it is easy to see that the formula

$$\mathop{\ast}_{i=1}^{n} F(S_i) \to F(S_0)$$

is provable in M and hence in L. Now, if $\mathbf{G} \vdash S_i$ for all $1 \leqslant i \leqslant n$, we have $\mathsf{L} \vdash F(S_i)$ which implies $\mathsf{L} \vdash \ast_{i=1}^{n} F(S_i)$ by the previous observation. Finally, since L is closed under modus ponens, $\mathsf{L} \vdash F(S_0)$ which implies $\mathbf{G} \vdash S_0$ by the completeness of $\mathbf{G}$. □

First, let us define the interpolation property for a sequent calculus.

Definition 4.5. *(essentially Maehara)* Let $\mathbf{G}$ and $\mathbf{H}$ be two sequent calculi.

- $\mathbf{G}$ has $\mathbf{H}$-*interpolation* if for any sequent S, and any partition $\Sigma \cup \Lambda$ of the antecedent of S, if S is provable in $\mathbf{G}$, then there exists a formula C such that
$$\mathbf{H} \vdash \Sigma \Rightarrow C \text{ and } \mathbf{H} \vdash \Lambda, C \Rightarrow S^s$$
and $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup S^s)$.

- $\mathbf{G}$ has *strong* $\mathbf{H}$*-interpolation* if for any sequent S, any partition $\Sigma \cup \Lambda$ of the antecedent of S, and any partition $\Theta \cup \Delta$ of the succedent of S, if S is provable in $\mathbf{G}$, then there exists a formula C such that

$$\mathbf{H} \vdash \Sigma \Rightarrow C, \Theta \text{ and } \mathbf{H} \vdash \Lambda, C \Rightarrow \Delta$$

and $V(C) \subseteq V(\Sigma \cup \Theta) \cap V(\Lambda \cup \Delta)$.

If $\mathbf{G}$ has strong $\mathbf{H}$-interpolation, then it also has $\mathbf{H}$-interpolation, by taking Θ as the empty multiset. If both Θ and Δ are non-empty multisets, then both $\mathbf{G}$ and $\mathbf{H}$ must be multi-conclusion sequent calculi. However, in the case that $\mathbf{G}$ has $\mathbf{H}$-interpolation, $\mathbf{G}$ and $\mathbf{H}$ can be either single- or multi-conclusion.

The following theorem shows that the interpolation property of a sequent calculus results the Craig interpolation of its logic.

Theorem 4.6. *Let* $\mathbf{G}$ *be a sequent calculus for the logic* L*. If* $\mathbf{G}$ *has the* $\mathbf{G}$*-interpolation property, then* L *has Craig interpolation.*

Proof. Let $\mathsf{L} \vdash \phi \to \psi$. Since $\mathbf{G}$ is a sequent calculus for L, by Definition 2.5 we have $\mathbf{G} \vdash \phi \Rightarrow \psi$. The sequent calculus $\mathbf{G}$ has the $\mathbf{G}$-interpolation property, therefore there exists θ such that $\mathbf{G} \vdash \phi \Rightarrow \theta$, $\mathbf{G} \vdash \theta \Rightarrow \psi$ and $V(\theta) \subseteq V(\phi) \cap V(\psi)$. Again, by Definition 2.5, $\mathsf{L} \vdash \phi \to \theta$ and $\mathsf{L} \vdash \theta \to \psi$ which completes the proof. ◻

The following theorem ensures that any subset of the focused axioms of a sequent calculus $\mathbf{H}$, has $\mathbf{H}$-interpolation property. It can also serve as an example to show how this notion of relative interpolation, Definition 4.5, works. The formula $\top^n$ is defined recursively. $\top^0$ is defined as 1 and $\top^{i+1} = \top^i * \top$.

Theorem 4.7. *Let* $\mathbf{G}$ *be a set of focused (context-free focused) axioms, and* $\mathbf{H}$ *be a sequent calculus containing* $\mathbf{G}$*. Then,*

- *if both* $\mathbf{G}$ *and* $\mathbf{H}$ *are single-conclusion and* $\mathbf{H}$ *extends* $\mathbf{FL_e}$ *(*$\mathbf{FL_e}^-$*),* $\mathbf{G}$ *has* $\mathbf{H}$*-interpolation, and*

- *if both* $\mathbf{G}$ *and* $\mathbf{H}$ *are multi-conclusion and* $\mathbf{H}$ *extends* $\mathbf{CFL_e}$ *(*$\mathbf{CFL_e}^-$*),* $\mathbf{G}$ *has strong* $\mathbf{H}$*-interpolation.*

Proof. We will only prove the case where **G** and **H** are both single-conclusion, using Definition 4.5. The proof for the multi-conclusion case is similar. Note that a sequent S is provable in **G** if it is one of the focused axioms. We will check each case separately. In the cases that the interpolant is a constant, i.e., $0, 1, \bot$, or $\top$, the condition on variables in Definition 4.5 is obviously satisfied since the set of variables of a constant is empty.

(1) In this case the sequent S is of the form $(\phi \Rightarrow \phi)$. For any partition Σ and Λ of the antecedent, we have to find a formula C such that $(\Sigma \Rightarrow C)$ and $(\Lambda, C \Rightarrow \phi)$ are provable in **H**. There are two cases to consider. First, if $\Sigma = \{\phi\}$ and $\Lambda = \{\}$. For this case define C to be ϕ. Obviously both conditions hold since we have $(\phi \Rightarrow \phi)$ as an axiom. Second, if $\Sigma = \{\}$ and $\Lambda = \{\phi\}$ define C as 1. We must show that $(\Rightarrow 1)$ and $(1, \phi \Rightarrow \phi)$ are provable in **H**. The former is an axiom of $\mathbf{FL_e^-}$ and hence provable in **H** since **H** extends $\mathbf{FL_e^-}$. The latter is the consequence of an instance of the rule $(1w)$ and the fact that $(\phi \Rightarrow \phi)$ is provable in **H**.

(2) For the case $(\Rightarrow \bar{\alpha})$, consider C to be 1. Then since both Σ and Λ are empty sequents, we must have $(\Rightarrow 1)$ and $(1 \Rightarrow \bar{\alpha})$ in **H**. The former is an axiom of $\mathbf{FL_e^-}$, and the latter is derived by applying the rule $(1w)$, which is again present in $\mathbf{FL_e^-}$, on $(\Rightarrow \bar{\alpha})$.

(3) For the axiom $(\bar{\beta} \Rightarrow)$, where $\bar{\beta} = \beta_1, \ldots, \beta_n$, there are three cases to consider:

  (i) If $\Lambda = \bar{\beta}$ and $\Sigma = \{\}$. Then define $C = 1$. It is clear that $\mathbf{H} \vdash \Sigma \Rightarrow 1$. Moreover, by the axiom and the rule $(1w)$ we will have $\mathbf{H} \vdash \Lambda, 1 \Rightarrow$.

  (ii) If $\Sigma = \bar{\beta}$ and $\Lambda = \{\}$, define $C = 0$. The reasoning is dual of the argument in (i).

  (iii) Otherwise, both Σ and Λ are non-empty. W.l.o.g. suppose $\Sigma = \beta_1, \ldots, \beta_i$ and $\Lambda = \beta_{i+1}, \ldots, \beta_n$, where $1 \leqslant i \leqslant n$. Define $C = \circledast \Sigma$. Then $\Sigma \Rightarrow C$ is provable in **H** by applying the rule $(R*)$ for $i - 1$ many times. Moreover, $(\Lambda, C \Rightarrow)$ holds in **H** by the axiom itself and applying the rule $(L*)$ for $i - 1$ many times. To check the condition on the variables, if $p \in V(C)$, then $p \in V(\Sigma)$. Recall that by Definition 3.6, each pair of the elements of $\bar{\beta}$ have the same set of variables. Since $\Sigma \cup \Lambda = \bar{\beta}$, and both Σ and Λ are non-empty, therefore, $p \in V(\Lambda)$. Hence $p \in V(\Sigma) \cap V(\Lambda)$.

(4) If S is of the form $\Gamma, \bar{\phi} \Rightarrow \Delta$, there are three cases to consider:

(i) If $\bar{\phi} \subseteq \Lambda$, define $C = \top$. Then $\Sigma \Rightarrow \top$ is an instance of an axiom in $\mathbf{FL_e}$, hence provable in $\mathbf{H}$. Moreover, since $\Gamma, \bar{\phi} \Rightarrow \Delta$ is an axiom in $\mathbf{G}$, and $\mathbf{H}$ extends $\mathbf{G}$, it is also provable in $\mathbf{H}$. Substitute $\{\top\} \cup \Lambda - \bar{\phi}$ for Γ to have $\top, \Lambda \Rightarrow \Delta$ provable in $\mathbf{H}$.

(ii) If $\bar{\phi} \subseteq \Sigma$, define $C = \bot$. Then, $\bot, \Lambda \Rightarrow \Delta$ is an instance of an axiom in $\mathbf{FL_e}$ and hence provable in $\mathbf{H}$. Moreover, substitute $\Sigma - \bar{\phi}$ for Γ and $\bot$ for Δ in $\Gamma, \bar{\phi} \Rightarrow \Delta$, to obtain $\Sigma \Rightarrow \bot$ provable in $\mathbf{H}$.

(iii) If none of the above happens, then both $\bar{\phi} \cap \Sigma$ and $\bar{\phi} \cap \Lambda$ are non-empty. Define $C = \ast(\Sigma \cap \bar{\phi}) * \top^n$ where n is the cardinal of $\Sigma - (\Sigma \cap \bar{\phi})$. First we have $\Sigma \Rightarrow C$ in $\mathbf{H}$. Because for any $\phi_i \in \Sigma \cap \bar{\phi}$, $\phi_i \Rightarrow \phi_i$ and for any $\psi \in \Sigma - (\Sigma \cap \bar{\phi})$ we have $\psi \Rightarrow \top$ (which is an instance of an axiom in $\mathbf{FL_e}$ and hence provable in $\mathbf{H}$), and at the end we use the rule $(R*)$ for appropriate many times. In the case that $\Sigma - (\Sigma \cap \bar{\phi}) = \varnothing$, then $\top^0 = 1$ and $\Rightarrow 1$ is provable in $\mathbf{H}$. Secondly, $\Lambda, C \Rightarrow \Delta$ is provable in $\mathbf{H}$. The reason is that the part of $\bar{\phi}$ which is occurred in Σ (and now in C) together with the part of $\bar{\phi}$ in Λ completes $\bar{\phi}$. Therefore, if we substitute the multiset $\{(\Lambda - \bar{\phi}), \top^n\}$ for Γ in the axiom $\Gamma, \bar{\phi} \Rightarrow \Delta$, we get $(\Lambda - \bar{\phi}), \top^n, (\Lambda \cap \bar{\phi}), (\Sigma \cap \bar{\phi}) \Rightarrow \Delta$, which after using the rule $(L*)$ for appropriate many times obtains $\Lambda, C \Rightarrow \Delta$. Finally, for the variables, if $\Sigma \cap \bar{\phi} = \varnothing$ then $V(C) = \varnothing$, as well. Otherwise, if $p \in V(C)$ then $p \in V(\Sigma \cap \bar{\phi})$. Since there is at least one of $\bar{\phi}$'s in Λ and each pair of the elements of $\bar{\phi}$ have the same variables, $p \in V(\Lambda)$ which completes the proof.

(5) If S is of the form $(\Gamma \Rightarrow \bar{\phi}, \Delta)$ define $C = \top$. Note that $\Sigma \Rightarrow \top$ is valid in $\mathbf{H}$ on the one hand and $C, \Lambda \Rightarrow \bar{\phi}, \Delta$ on the other. The latter is an instance of the axiom itself and hence valid.

Note that in the context-free axioms, $1, 2$, and 3, we only made use of the fact that $\mathbf{H}$ extends $\mathbf{FL_e^-}$. In 4 and 5, we used the axioms for $\bot$ and $\top$, which was possible since $\mathbf{H}$ extends $\mathbf{FL_e}$ in these cases. □

4.1 The Single-conclusion Case

Now we are ready to prove that semi-analytic rules respect the interpolation property. This subsection is devoted to the single-conclusion case. Throughout the rest of Section 4, we will assume that $1 \leqslant i \leqslant n, 1 \leqslant r \leqslant u_i, 1 \leqslant j \leqslant m$, and $1 \leqslant s \leqslant v_j$. However, for simplicity, we omit these domains. Moreover, we use the convention that $\{\Gamma'_i\}_{i=1}^n = \Gamma'$ and $\{\Pi'_i\}_{j=1}^m = \Pi'$. The same goes for Γ''_i, Π''_j, Δ_i, Δ'_i, and Δ''_i.

Theorem 4.8. *Let* **G** *and* **H** *be two single-conclusion sequent calculi such that* **H** *extends* $\mathbf{FL_e}^-$ *and satisfies the modal admissibility conditions. Suppose* **H** *is an axiomatic extension of* **G***, to which we add the "single-conclusion" version of any of the following rules:*

- (i) *semi-analytic rules;*
- (ii) *any of the rules* (K), (D), $(L$S4), *or* $(R$S4)*;*
- (iii) *any of the rules* (4) *or* (4D)*, given the condition that the left weakening rule for boxed formulas is admissible in* **H***;*
- (iv) *context-sharing semi-analytic rules, given the condition that the left and right weakening rules and the left context-sharing implication rule are all admissible in* **H***.*

Then, if **G** *has* **H***-interpolation, so does* **H***.*

Proof. In each case, we will show that **H** has **H**-interpolation. Let S be a sequent of the form $\Gamma \Rightarrow \Delta$ provable in **H** with the proof π. We will use induction on the **H**-length of π. If the **H**-length is zero, then it means that the proof is in **G** and hence the existence of the interpolant is guarantied by the assumption. For the induction step, we will investigate the last rule used in the proof. Since the proofs of most of the cases are similar, we will only prove some of them.

(i) There are two possibilities: either the added rule is a left semi-analytic rule, or it is a right one.

- Consider the case where the last rule used in the proof is a left semi-analytic rule and the main formula, ϕ, is in part Λ in the Definition 4.5 (or informally, ϕ appears in the same sequent as Δ does). Hence, we are in the case that S^a is partitioned as $\Sigma = \{\Gamma', \Pi'\}$ and $\Lambda = \{\Gamma'', \Pi'', \phi\}$ and S is of the form $(\Gamma', \Gamma'', \Pi', \Pi'', \phi \Rightarrow \Delta)$. We have to find a formula C that satisfies $(\Gamma', \Pi' \Rightarrow C)$ and $(\Gamma'', \Pi'', \phi, C \Rightarrow \Delta)$. Since the last rule used in the proof is a left semi-analytic one, it is of the form

$$\frac{\langle\langle \Pi'_j, \Pi''_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Pi', \Pi'', \Gamma', \Gamma'', \phi \Rightarrow \Delta} \ (\dagger)$$

Using the induction hypothesis for the premises, there are C_{js} and D_{ir} for each i, j, r, and s such that

$$\Pi'_j \Rightarrow C_{js} \quad , \quad \Pi''_j, \bar{\psi}_{js}, C_{js} \Rightarrow \bar{\theta}_{js}$$

$$\Gamma'_i \Rightarrow D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \Delta_i$$

Using the rule $(R\wedge)$ for the left sequents and the rule $(L\wedge)$ for the right ones, we get

$$\Pi'_j \Rightarrow \bigwedge_s C_{js} \quad , \quad \Pi''_j, \bar{\psi}_{js}, \bigwedge_s C_{js} \Rightarrow \bar{\theta}_{js}$$

$$\Gamma'_i \Rightarrow \bigwedge_r D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \Delta_i$$

For the left sequents, use the rule $(R*)$ to obtain

$$\Pi'_1, \ldots, \Pi'_m, \Gamma'_1, \ldots, \Gamma'_n \Rightarrow (\mathop{\ast}_j \bigwedge_s C_{js}) * (\mathop{\ast}_i \bigwedge_r D_{ir}).$$

And, apply the rule $(\dagger)$ on the right sequents, such that we substitute each $(\Pi''_j, \bigwedge_s C_{js})$ for Π''_j and each $(\Gamma''_i, \bigwedge_r D_{ir})$ for Γ''_i in $(\dagger)$. This is possible, first because of the condition in definition of semi-analytic rules (Definition 3.2) that contexts are free for any substitution of multisets and second because $\bigwedge_s C_{js}$ depends only on j and not on s and $\bigwedge_r D_{ir}$ depends only on i and not on r. Then, using the rule $(L*)$ on the conclusion of this rule, we obtain

$$\Pi''_1, \ldots, \Pi''_m, \Gamma''_1, \ldots, \Gamma''_n, (\mathop{\ast}_j \bigwedge_s C_{js}) * (\mathop{\ast}_i \bigwedge_r D_{ir}), \phi \Rightarrow \Delta_1, \ldots, \Delta_n$$

Therefore, we let C be $(\circledast_j \bigwedge_s C_{js}) * (\circledast_i \bigwedge_r D_{ir})$ and we have proved $(\Gamma', \Pi' \Rightarrow C)$ and $(\Gamma'', \Pi'', \phi, C \Rightarrow \Delta)$.

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is in one of C_{js} or D_{ir}. If it is in C_{js}, by induction hypothesis, it is either in Π'_j (which means it is in Σ), or it is in $\{\Pi''_j, \bar{\psi}_{js}, \bar{\theta}_{js}\}$. If it is in Π''_j, then it is in Λ and if it is in either $\bar{\psi}_{js}$ or $\bar{\theta}_{js}$, since the rule is occurence preserving, it also appears in ϕ which means it appears in Λ.
If the atom is in D_{ir}, we reason in a similar way, and it either appears in Γ'_i (and hence in Σ) or it appears in $\{\Gamma''_i, \bar{\phi}_{ir}, \Delta_i\}$ and hence in $\Lambda \cup \Delta$.

- Consider the case where the last rule used in the proof is a left semi-analytic rule and the main formula, ϕ, is this time in Σ in the Definition 4.5. Hence, S^a is partitioned as $\Sigma = \{\Gamma', \Pi', \phi\}$ and $\Lambda = \{\Gamma'', \Pi''\}$. The sequent S is again of the form $(\Gamma', \Gamma'', \Pi', \Pi'', \phi \Rightarrow \Delta)$ and we have to find a formula C that satisfies $(\Gamma', \Pi', \phi \Rightarrow C)$ and $(\Gamma'', \Pi'', C \Rightarrow \Delta)$. Since S is a single-conclusion sequent, and $\Delta = \Delta_1, \ldots, \Delta_n$, at most one of Δ_i's can be non-empty. W.l.o.g., suppose that for $i \neq 1$ we have $\Delta_i = \varnothing$ and $\Delta_1 = \Delta$. Therefore, the last rule used in the proof is of the form

$$\frac{\langle\langle \Pi'_j, \Pi''_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s\rangle_j \qquad \langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma'_1, \Gamma''_1, \bar{\phi}_{1r} \Rightarrow \Delta \rangle_r}{\Pi', \Pi'', \Gamma', \Gamma'', \phi \Rightarrow \Delta} \; (\ddagger)$$

Using the induction hypothesis for the premises, there exist formulas C_{js} and D_{ir} for each $i \neq 1$ and j, s, r and formulas D_{1r} for each r such that

$$\Pi'_j, \bar{\psi}_{js}, C_{js} \Rightarrow \bar{\theta}_{js} \quad , \quad \Pi''_j \Rightarrow C_{js}$$

$$\Gamma'_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow D_{1r} \quad , \quad \Gamma''_1, D_{1r} \Rightarrow \Delta$$

Using the rules $(L\wedge)$, $(R\wedge)$, $(R\vee)$ and $(L\vee)$, we have (for $i \neq 1$)

$$\Pi'_j, \bar{\psi}_{js}, \bigwedge_s C_{js} \Rightarrow \bar{\theta}_{js} \quad , \quad \Pi''_j \Rightarrow \bigwedge_s C_{js}$$

$$\Gamma'_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow \bigwedge_r D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r D_{1r} \quad , \quad \Gamma''_1, \bigvee_r D_{1r} \Rightarrow \Delta$$

If we substitute the above left sequents in the original rule (‡), such that we take each $(\Pi'_j, \bigwedge_s C_{js})$ as Π'_j, each $(\Gamma'_i, \bigwedge_r D_{ir})$ as Γ'_i (for $i \neq 1$), and Γ'_1 as Γ'_1, we get (for $i \neq 1$)

$$\Pi', \Gamma', \bigwedge_s C_{js}, \bigwedge_r D_{ir}, \phi \Rightarrow \bigvee_r D_{1r}.$$

And first, using the rule $(L*)$ and then $(R \rightarrow)$ we get

$$\Pi', \Gamma', \phi \Rightarrow (\mathop{\circledast}_{i \neq 1} \bigwedge_r D_{ir}) * (\mathop{\circledast}_j \bigwedge_s C_{js}) \rightarrow \bigvee_r D_{1r}.$$

On the other hand, using the rules $(R*)$ and $(L \rightarrow)$ for the right sequents we have

$$\Pi'', \Gamma'', (\mathop{\circledast}_{i \neq 1} \bigwedge_r D_{ir}) * (\mathop{\circledast}_j \bigwedge_s C_{js}) \rightarrow \bigvee_r D_{1r} \Rightarrow \Delta$$

It is enough to take C as $(\mathop{\circledast}_{i \neq 1} \bigwedge_r D_{ir}) * (\mathop{\circledast}_j \bigwedge_s C_{js}) \rightarrow \bigvee_r D_{1r}$ to finish the proof of this case.

It is easy to check the condition on the variables of C (similar to the previous case). To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{js} or D_{ir} for $(i \neq 1)$ or in D_{1r}. By induction hypothesis if it is in C_{js}, it is both in $\{\Pi'_j, \bar{\psi}_{js}, \bar{\theta}_{js}\}$ and in Π''_j. If it is in D_{ir} for $(i \neq 1)$, then it is both in $\{\Gamma'_i, \bar{\phi}_{ir}\}$ and in Γ''_i. And if it is in D_{1r}, then it is both in $\{\Gamma'_1, \bar{\phi}_{1r}\}$ and in $\{\Gamma''_1, \Delta\}$. One can easily check that therefore, the atom will be both in $\Sigma = \{\Gamma', \Pi', \phi\}$ and in $\Lambda \cup \Delta = \{\Gamma'', \Pi'', \Delta\}$. Note that in the reasoning we will need the occurrence preserving property, as well.

- The case where the last rule used in the proof is a right semi-analytic rule, is similar. Consider the case where the last rule used in the proof is a right semi-analytic rule. Since $S = (\Gamma \Rightarrow \Delta)$ is a single-conclusion sequent, $\Delta = \phi$. Now, for the partitions $\Sigma = \Gamma''$ and $\Lambda = \Gamma'$ of S^a, we have to find a formula C that satisfies $(\Gamma'' \Rightarrow C)$ and $(\Gamma', C \Rightarrow \phi)$. Therefore, the last rule used in the proof must have been of the form

  $$\frac{\langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma', \Gamma'' \Rightarrow \phi} \ (\dagger\ddagger)$$

  Using the induction hypothesis, we get

  $$\Gamma'_i, C_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \quad , \quad \Gamma''_i \Rightarrow C_{ir}.$$

  Using the rules $(L\wedge)$ and $(R\wedge)$ we have

  $$\Gamma'_i, \bigwedge_r C_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \quad , \quad \Gamma''_i \Rightarrow \bigwedge_r C_{ir}.$$

  Substituting the left sequents for each i and r in the original rule $(\dagger\ddagger)$ and then using the rule $(L*)$, we conclude

  $$\Gamma', \mathop{\ast}_i (\bigwedge_r C_{ir}) \Rightarrow \phi.$$

  On the other hand, using the rule $(R*)$ for the sequents $\Gamma''_i \Rightarrow \bigwedge_r C_{ir}$ for $1 \leqslant i \leqslant n$, we get $\Gamma'' \Rightarrow \mathop{\ast}_i (\bigwedge_r C_{ir})$ which means that the formula $\mathop{\ast}_i (\bigwedge_r C_{ir})$ serves as the interpolant C.

  To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{ir}. Then by induction hypothesis it is both in $\{\Gamma'_i, \bar{\phi}_{ir}, \bar{\psi}_{ir}\}$ and in Γ''_i. It is easy to check that it meets the conditions needed.

(ii) Consider the case where the last rule used in the proof is either K or D, and if it is D, then the rule K is admissible in $\mathbf{H}$. Then, the sequent S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\Delta$, where Δ is either one formula (in the case that the last rule is K) or it is the empty sequence (in the case that the last rule is D). Recall that $\Box\varnothing = \varnothing$. We have to find a formula C that satisfies $\Box\Gamma' \Rightarrow C$ and $C, \Box\Gamma'' \Rightarrow \Box\Delta$. The last rule used in the proof is of the form

$$\frac{\Gamma', \Gamma'' \Rightarrow \Delta}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\Delta}$$

Using the induction hypothesis there exists a formula D such that

$$\Gamma' \Rightarrow D \quad , \quad \Gamma'', D \Rightarrow \Delta.$$

Then, using the rule K for both of them (or if $\Delta = \varnothing$, use the rule D for the sequent $(\Gamma'', D \Rightarrow)$), we get

$$\Box\Gamma' \Rightarrow \Box D \quad , \quad \Box\Gamma'', \Box D \Rightarrow \Box\Delta.$$

It is worth mentioning that, as observed above, when we are dealing with the rule D, the rule K must be present in **H**, so that we would be able to obtain $\Box\Gamma'', \Box D \Rightarrow \Box\Delta$ from $\Gamma' \Rightarrow D$ in the calculus. Let $\Box D$ be the formula C and we are done. And since $V(D) \subseteq V(\Gamma') \cap V(\Gamma'' \cup \Delta)$ we have $V(C) \subseteq V(\Box\Gamma') \cap V(\Box\Gamma'' \cup \Box\Delta)$, because the set of variables of $\Box\Pi$ for a multiset Π is the same as the one for Π.

For the other rules, note that the rule $(L\mathsf{S4})$ is of the from of a left single-conclusion semi-analytic rule. Therefore, by (i), we can freely add it to **H**. The proof of the case of the rule $(R\mathsf{S4})$ is similar to the case (ii). Now, suppose the last rule used in the proof is the rule $(R\mathsf{S4})$, and $(L\mathsf{S4})$ is also present in **H**. Therefore, for the sequent S of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$, we have to find a formula C that satisfies $\Box\Gamma' \Rightarrow C$ and $C, \Box\Gamma'' \Rightarrow \Box\phi$. The last rule used in the proof is of the form

$$\frac{\Box\Gamma', \Box\Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis there exists a formula D such that

$$\Box\Gamma' \Rightarrow D \quad , \quad \Box\Gamma'', D \Rightarrow \phi.$$

On the left sequent, apply the rule $(R\mathsf{S4})$. On the right sequent, first apply the rule $(L\mathsf{S4})$ and then the rule $(R\mathsf{S4})$. We get

$$\Box\Gamma' \Rightarrow \Box D \quad , \quad \Box\Gamma'', \Box D \Rightarrow \Box\phi$$

It is easy to see that $C = \Box D$ works in this case.

(iii) This case is similar to the case (ii). Now, consider the case that the last rule used in the proof is $\mathsf{4}$. Then, the sequent S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$, and we have to find a formula C that satisfies $\Box\Gamma' \Rightarrow C$ and $C, \Box\Gamma'' \Rightarrow \Box\phi$. The last rule used in the proof is of the form

$$\frac{\Gamma', \Gamma'', \Box\Gamma', \Box\Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis there exists a formula D such that

$$\Gamma', \Box\Gamma' \Rightarrow D \quad , \quad \Gamma'', \Box\Gamma'', D \Rightarrow \phi.$$

If we use the rule $\mathsf{4}$ on the left sequent and using the left weakening rule on the right sequent (adding $\Box D$ to the left hand side of the sequent) and then using the rule $\mathsf{4}$, we get

$$\Box\Gamma' \Rightarrow \Box D \quad , \quad \Box\Gamma'', \Box D \Rightarrow \Box\phi$$

If we take $C = \Box D$, then the claim follows. Checking the atoms is similar as before.
For the proof of the case $\mathsf{4D}$ is identical to the proof of the rule $\mathsf{4}$, if we ignore ϕ and $\Box\phi$ everywhere.

(iv) The proof of this case is similar to (i) where the added rule is a left semi-analytic rule. Finally, we will investigate the case where the last rule used in the proof is a context-sharing semi-analytic one. There are two cases to consider, based on the appearance of the main formula in the partition of S^a.

- Suppose the main formula, ϕ, is in Λ in Definition 4.5 (or informally, ϕ appears in the same sequent as Δ does). Hence, S^a is partitioned as $\Sigma = \{\Gamma'\}$ and $\Lambda = \{\Gamma'', \phi\}$, and the sequent S is of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta)$. We have to find a formula C that satisfies $(\Gamma' \Rightarrow C)$ and $(\Gamma'', \phi, C \Rightarrow \Delta)$. Therefore, the last rule used in the proof is of the form

$$\frac{\langle\langle \Gamma'_i, \Gamma''_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is} \rangle_s \rangle_i \qquad \langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Gamma', \Gamma'', \phi \Rightarrow \Delta} \; (\bigstar)$$

  Using the induction hypothesis for the premises, there are formulas C_{is} and D_{ir} for each i, r, and s such that

$$\Gamma'_i \Rightarrow C_{is} \quad , \quad \Gamma''_i, \bar{\psi}_{is}, C_{is} \Rightarrow \bar{\theta}_{is}$$

$$\Gamma'_i \Rightarrow D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \Delta_i$$

Using the rules $(R\wedge)$ and $(L\wedge)$ we have

$$\Gamma'_i \Rightarrow \bigwedge_s C_{is} \quad , \quad \Gamma''_i, \bar{\psi}_{is}, \bigwedge_s C_{is} \Rightarrow \bar{\theta}_{is}$$

$$\Gamma'_i \Rightarrow \bigwedge_r D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \Delta_i$$

To be able to apply the original rule $(\bigstar)$ on the right sequents above, we have to make sure that we can make the contexts in the antecednts become equivalent. Therefore, what we do is using the rule $(L\wedge)$ to get

$$\Gamma''_i, \bar{\psi}_{is}, (\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir}) \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, (\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is}) \Rightarrow \Delta_i.$$

Now, they have the same contexts in the antecedents and we can substitute them in the original rule and conclude

$$\Gamma'', \langle (\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is}) \rangle_i, \phi \Rightarrow \Delta.$$

And, using the rule $(L*)$ we get

$$\Gamma'', \mathop{\circledast}_i [(\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})], \phi \Rightarrow \Delta.$$

On the other hand, considering the sequents $(\Gamma'_i \Rightarrow \bigwedge_s C_{is})$ and $(\Gamma'_i \Rightarrow \bigwedge_r D_{ir})$, if for each i we use the rule $(R\wedge)$, we get

$$\Gamma'_i \Rightarrow (\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is}),$$

and then using the rule $(R*)$ we have

$$\Gamma' \Rightarrow \circledast_i [(\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})].$$

Then, we can see that $\circledast_i [(\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})]$ serves as C.

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{is} or D_{ir}. By induction hypothesis, if it is in C_{is}, then it is both in Γ'_i and in $\{\Gamma''_i, \bar{\psi}_{is}, \bar{\theta}_{is}\}$ and if it is in D_{ir}, then it is both in Γ'_i and in $\{\Gamma''_i, \bar{\phi}_{ir}, \Delta_i\}$. It is easy to check that it meets the condition for variables.

- Now, suppose the main formula, ϕ, is in Σ in Definition 4.5. Hence, S^a is partitioned as $\Sigma = \{\Gamma', \phi\}$ and $\Lambda = \{\Gamma''\}$ and the sequent S is of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta)$. We have to find a formula C that $(\Gamma', \phi \Rightarrow C)$ and $(\Gamma'', C \Rightarrow \Delta)$. W.l.o.g., suppose that for $i \neq 1$ we have $\Delta_i = \varnothing$ and $\Delta_1 = \Delta$. Therefore, the last rule used in the proof is of the form

$$\frac{\langle\langle \Gamma'_i, \Gamma''_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is} \rangle_s \rangle_i \qquad \langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \rangle_r \rangle_{i \neq 1} \qquad \langle \Gamma'_1, \Gamma''_1, \bar{\phi}_{1r} \Rightarrow \Delta \rangle_r}{\Gamma', \Gamma'', \phi \Rightarrow \Delta} \; (\star)$$

Using the induction hypothesis for the premises, there exist formulas C_{is} and D_{ir} such that

$$\Gamma'_i, \bar{\psi}_{is}, C_{is} \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma''_i \Rightarrow C_{is}$$

$$\Gamma'_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow D_{ir} \;\; (\text{for } i \neq 1)$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow D_{1r} \quad , \quad \Gamma''_1, D_{1r} \Rightarrow \Delta$$

Using the rules $(L\wedge)$, $(R\wedge)$, $(R\vee)$ and $(L\vee)$, we have (for $i \neq 1$)

$$\Gamma'_i, \bar{\psi}_{is}, \bigwedge_s C_{is} \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma''_i \Rightarrow \bigwedge_s C_{is}$$

$$\Gamma'_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow \bigwedge_r D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r D_{1r} \quad , \quad \Gamma''_1, \bigvee_r D_{1r} \Rightarrow \Delta$$

$$\Gamma'_1, \bar{\psi}_{1s}, \bigwedge_s C_{1s} \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma''_1 \Rightarrow \bigwedge_s C_{1s}$$

To be able to apply the original rule ($\star$) on the left sequents above, we have to make sure that we can make the contexts in the antecednts become equivalent. For ($i \neq 1$) use the rule ($L\wedge$) to obtain the context $\{\Gamma'_i, (\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})\}$ and for ($i = 1$) use the left weakening rule (on the left sequent in the third row) to get the context $\{\Gamma'_1, \bigwedge_s C_{1s}\}$. If we substitute the updated left sequents in the original rule ($\star$), we get

$$\Gamma', \langle(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})\rangle_{i\neq 1}, \bigwedge_s C_{1s}, \phi \Rightarrow \bigvee_r D_{1r}.$$

If we first apply the rule ($L*$) and then ($R \to$) on the above sequent, we get

$$\Gamma', \phi \Rightarrow (\mathop{\circledast}_{i\neq 1}[(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] * \bigwedge_s C_{1s}) \to \bigvee_r D_{1r}.$$

On the other hand, applying the rule ($R\wedge$) on the sequents $\Gamma''_i \Rightarrow \bigwedge_s C_{is}$ and $\Gamma''_i \Rightarrow \bigwedge_r D_{ir}$, for ($i \neq 1$), we get $\Gamma''_i \Rightarrow (\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})$. Together with the sequent $\Gamma''_1 \Rightarrow \bigwedge_s C_{1s}$, and using the rule ($R*$) we get

$$\Gamma'' \Rightarrow (\mathop{\circledast}_{i\neq 1}[(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] * \bigwedge_s C_{1s}).$$

Moreover, we have $\Gamma''_1, \bigvee_r D_{1r} \Rightarrow \Delta$. Apply the left weakening rule on it to get $\Gamma'', \bigvee_r D_{1r} \Rightarrow \Delta$. Now, we can use the context-sharing left implication rule to get

$$\Gamma'', (\mathop{\circledast}_{i\neq 1}[(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] * \bigwedge_s C_{1s}) \to \bigvee_r D_{1r} \Rightarrow \Delta.$$

We can see that $(\mathop{\circledast}_{i\neq 1}[(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] * \bigwedge_s C_{1s}) \to \bigvee_r D_{1r}$ serves as C and we are done.

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{is} or D_{ir}. By induction hypothesis, if it is in C_{is}, then it is both in $\{\Gamma'_i, \bar{\psi}_{is}, \bar{\theta}_{is}\}$ and in Γ''_i and if it is in D_{ir} for $(i \neq 1)$, then it is both in $\Gamma'_i, \bar{\phi}_{ir}$, and in $\{\Gamma''_i\}$. If it is in D_{1r}, then it is both in $\Gamma'_1, \bar{\phi}_{1r}$, and in $\{\Gamma''_1, \Delta\}$. It is easy to check that it meets the conditions on variables.

□

4.2 The Multi-conclusion Case

In this subsection we will generalize Theorem 4.8 to also cover the multi-conclusion case.

Theorem 4.9. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be two multi-conclusion sequent calculi such that* $\mathbf{H}$ *extends* $\mathbf{CFL_e}^-$ *and satisfies the modal admissibility conditions. Suppose* $\mathbf{H}$ *is an axiomatic extension of* $\mathbf{G}$*, to which we add the "multi-conclusion" version of any of the following rules:*

- (i) *semi-analytic rules;*
- (ii) *any of the rules* (K), (D), (LS4)*, or* (RS4)*;*
- (iii) *any of the rules* (4) *or* (4D)*, given the condition that the left weakening rule for boxed formulas is admissible in* $\mathbf{H}$*.*

Then, if $\mathbf{G}$ *has strong* $\mathbf{H}$*-interpolation, so does* $\mathbf{H}$*.*

Proof. The proof is similar to the proof of Theorem 4.8. First, note that in the multi-conclusion case, a context-charing semi-analytic rule is also of the form of a left semi-analytic rule. Therefore, the case of adding a context-charing semi-analytic rule to $\mathbf{H}$ is covered in (i). The proof again uses induction on the $\mathbf{H}$-length of π, where π is a proof of a sequent S in $\mathbf{H}$.

(i) We add to $\mathbf{H}$ a multi-conclusion semi-analytic rule. There are four cases to consider, depending on whether the rule is a left or a right one and also on the appearance of the main formula in the partitions of S^a and S^s. We will only prove one of these cases.

∘ Consider the case where the last rule used in the proof is a left multi-conclusion semi-analytic rule and the main formula, ϕ, is in Λ in (the second part of) Definition 4.5. Hence, the sequent S is of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta'')$ and we have to find a formula C that satisfies $(\Gamma' \Rightarrow C, \Delta')$ and $(\Gamma'', \phi, C \Rightarrow \Delta'')$. The last rule used in the proof is of the form

$$\frac{\langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta'_i, \Delta''_i \rangle_r \rangle_i}{\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta''} \; (\bigstar)$$

By induction hypothesis for the premises, for every i and r there exists a formula C_{ir} such that

$$\Gamma'_i \Rightarrow C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, C_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta''_i.$$

Using the rule $(R\wedge)$ and $(L\wedge)$ we have for every i

$$\Gamma'_i \Rightarrow \bigwedge_r C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, \bigwedge_r C_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta''_i.$$

Using the rule $(R*)$ for the left sequents we get

$$\Gamma' \Rightarrow \mathop{\circledast}_i \bigwedge_r C_{ir}, \Delta'.$$

Moreover, if we substitute the right sequents in the original rule $(\bigstar)$, and then using the rule $(L*)$, we get

$$\Gamma'', \phi, \mathop{\circledast}_i \bigwedge_r C_{ir} \Rightarrow \Delta''$$

Hence, we take C as $\mathop{\circledast}_i \bigwedge_r C_{ir}$ and we are done.

To check $V(C) \subseteq V(\Gamma' \cup \Delta') \cap V(\{\Gamma'' \cup \{\phi\}\} \cup \Delta'')$, note that an atom is in C if and only if it is in one of C_{ir}'s. Then, by induction hypothesis, it is in $(\Gamma'_i \cup \Delta'_i)$ and in $\{\Gamma''_i, \bar{\phi}_{ir}, \bar{\psi}_{ir}, \Delta''_i\}$. It can be easily seen that the claim holds; the only thing to remember is that if the atom is in either $\bar{\phi}_{ir}$ or in $\bar{\psi}_{ir}$, since the rule is occurrence preserving, it also appears in ϕ.

- Consider the case where the last rule used in the proof is a left multi-conclusion semi-analytic rule and the main formula, ϕ, is in Σ in the Definition 4.5. Hence, the sequent S is of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta'')$ and we have to find a formula C that satisfies $(\Gamma', \phi \Rightarrow C, \Delta')$ and $(\Gamma'', C \Rightarrow \Delta'')$. The last rule used in the proof is of the form

$$\frac{\langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta'_i, \Delta''_i \rangle_r \rangle_i}{\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta''} \; (\dagger)$$

By induction hypothesis for the premises, for every i and r there exists a formula C_{ir} such that

$$\Gamma'_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, C_{ir} \Rightarrow \Delta''_i.$$

Using the rules $(R\vee)$ and $(L\vee)$, we have for every i

$$\Gamma'_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bigvee_r C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, \bigvee_r C_{ir} \Rightarrow \Delta''_i.$$

If we substitute the left sequents above in the original rule (†), we get

$$\Gamma'_1, \ldots, \Gamma'_n, \phi \Rightarrow \bigvee_r C_{1r}, \ldots, \bigvee_r C_{nr}, \Delta'_1, \ldots, \Delta'_n.$$

Using the convention (stated at the beginning of Subsection 4.1) and applying the rule $(R+)$ we get

$$\Gamma', \phi \Rightarrow \bigplus_i \bigvee_r C_{ir}, \Delta'.$$

On the other hand, applying the rule $(L+)$ on the sequents $\Gamma''_i, \bigvee_r C_{ir} \Rightarrow \Delta''_i$ we obtain

$$\Gamma'', \bigplus_i \bigvee_r C_{ir} \Rightarrow \Delta''.$$

It is enough to take C as $\bigplus_i \bigvee_r C_{ir}$ to finish the proof of this case.

To check $V(C) \subseteq V(\{\Gamma' \cup \{\phi\}\} \cup \Delta') \cap V(\Gamma'' \cup \Delta'')$, note that an atom is in C if and only if it is in one of C_{ir}'s. Then, by induction hypothesis, it is in $\{\Gamma'_i, \bar{\phi}_{ir}, \bar{\psi}_{ir}, \Delta'_i\}$ and in $(\Gamma''_i \cup \Delta''_i)$. It can be easily seen that the claim holds; the only thing to remember is that if the atom is in either $\bar{\phi}_{ir}$ or in $\bar{\psi}_{ir}$, since the rule is occurrence preserving, it also appears in ϕ.

Let the last rule used in the proof be the rule K. Therefore, the sequent S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$. There can be two cases based on the appearance of $\Box\phi$ in the partition of S^s. Either, in Definition 4.5, $\Delta = \Box\phi$ and $\Theta = \varnothing$ and we have to show that there exists a formula C such that $\Box\Gamma' \Rightarrow C$ and $\Box\Gamma'', C \Rightarrow \Box\phi$ are provable in $\mathbf{H}$. Or $\Theta = \Box\phi$ and $\Delta = \varnothing$ in Definition 4.5 and we have to show that there exists C such that $\Box\Gamma' \Rightarrow C, \Box\phi$ and $\Box\Gamma'', C \Rightarrow$ hold in $\mathbf{H}$. Since the proof of the first case is similar to the proof in Theorem 4.8, we will investigate the second case. The last rule used in the proof is of the form

$$\frac{\Gamma', \Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis for the premise, there exists a formula D such that the following are provable in $\mathbf{H}$

$$\Gamma' \Rightarrow D, \phi \quad , \quad D, \Gamma'' \Rightarrow$$

Apply the rule $(L \to)$ on the axiom $(\Rightarrow 0)$ together with the above sequent on the left. Moreover, apply the rule $(0w)$ on the sequent above on the right and then use the rule $(R \to)$. As the result we get

$$\Gamma', \neg D \Rightarrow, \phi \quad , \quad \Gamma'' \Rightarrow \neg D.$$

Use the rule K on both sequents to obtain

$$\Box\Gamma', \Box\neg D \Rightarrow \Box\phi \quad , \quad \Box\Gamma'' \Rightarrow \Box\neg D.$$

It is easy to see that from the above sequents we can get

$$\Box\Gamma' \Rightarrow \neg\Box\neg D, \Box\phi \quad , \quad \neg\Box\neg D, \Box\Gamma'' \Rightarrow$$

which means we have to take $C = \neg\Box\neg D$. It is also easy to check the condition on the variables of the interpolant.
The rest of the cases are proved similarly to the proof of Theorem 4.8. As an example, let us investigate one case. Consider the case where the last rule used in the proof is the rule $\mathsf{4}$, and we assume that the left weakening rule for boxed formulas is admissible in **H**. Therefore, the sequent S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$. There can be two cases based on the appearance of $\Box\phi$ in the partition of S^s. Either, in Definition 4.5, $\Delta = \Box\phi$ and $\Theta = \varnothing$ and we have to show that there exists a formula C such that $\Box\Gamma' \Rightarrow C$ and $\Box\Gamma'', C \Rightarrow \Box\phi$ are provable in **H**. Or $\Theta = \Box\phi$ and $\Delta = \varnothing$ in Definition 4.5 and we have to show that there exists C such that $\Box\Gamma' \Rightarrow C, \Box\phi$ and $\Box\Gamma'', C \Rightarrow$ hold in **H**. Since the proof of the first case is similar to the proof in Theorem 4.8, we will investigate the second case. The last rule used in the proof is of the form

$$\frac{\Gamma', \Gamma'', \Box\Gamma', \Box\Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis for the premise, there exists a formula D such that the following hold in **H**

$$\Gamma', \Box\Gamma' \Rightarrow D, \phi \quad , \quad D, \Gamma'', \Box\Gamma'' \Rightarrow$$

Apply the rule $(L \to)$ on the axiom $(\Rightarrow 0)$ together with the sequent above on the left. Moreover, apply the rule $(0w)$ on the above sequent on the right and then use the rule $(R \to)$. As the result, we get

$$\Gamma', \Box\Gamma', \neg D \Rightarrow \phi \quad , \quad \Gamma'', \Box\Gamma'' \Rightarrow \neg D.$$

Use the left weakening rule for boxed formulas on the left sequent (to add $\Box\neg D$ to the antecedent of the sequent). Then, apply the rule $\mathsf{4}$ on both sequents to get

$$\Box\Gamma', \Box\neg D \Rightarrow \Box\phi \quad , \quad \Box\Gamma'' \Rightarrow \Box\neg D.$$

It is easy to see that from the above sequents we can get

$$\Box\Gamma' \Rightarrow \neg\Box\neg D, \Box\phi \quad , \quad \neg\Box\neg D, \Box\Gamma'' \Rightarrow$$

Take $C = \neg\Box\neg D$ as the interpolant. It is easy to check the condition for the variables of C.

The cases where the last rule used in the proof is the rule D or $\mathsf{4D}$ is similar to the proof of the same cases in Theorem 4.8. In the case of the rule $(R\mathsf{S4})$, we have exactly the same cases as in the rule K:

$$\Box\Gamma' \Rightarrow C \quad , \quad \Box\Gamma'', C \Rightarrow \Box\phi$$

and

$$\Box\Gamma' \Rightarrow C, \Box\phi \quad , \quad \Box\Gamma'', C \Rightarrow$$

Only the second case is new (the proof for the first one is the same as the proof of the same case in Theorem 4.8). The proof of the second case is the same as the case for the rule K in the above, and $C = \neg\Box\neg D$ works here, as well. The cases where the last rule in the proof is a right multi-conclusion semi-analytic one is similar and we do not investigate them here. $\Box$

For the rest of this section, define $\mathcal{R}$ as the set of the modal rules $\{\mathsf{K}, \mathsf{D}, \mathsf{4}, \mathsf{4D}, R\mathsf{S4}, L\mathsf{S4}\}$. Combining Theorems 4.7, 4.8 and 4.9 we will have:

Theorem 4.10. *Let* $\mathbf{H}$ *be a sequent calculus that satisfies the modal admissibility conditions.*

- (i) *If* $\mathbf{H}$ *is single-conclusion and is an extension of* $\mathbf{FL_e}$ *(*$\mathbf{FL_e}^-$*) consisting of focused axioms (context-free focused axioms), semi-analytic rules, and any subset of* $\mathcal{R}$*, then* $\mathbf{H}$ *has* $\mathbf{H}$*-interpolation.*

- (ii) *If* $\mathbf{H}$ *is single-conclusion and is an extension of* $\mathbf{LJ}$ *consisting of focused axioms, semi-analytic rules, context-sharing semi-analytic rules, and any subset of* $\mathcal{R}$*, then* $\mathbf{H}$ *has* $\mathbf{H}$*-interpolation.*

(iii) *If* $\mathbf{H}$ *is multi-conclusion and is an extension of* $\mathbf{CFL_e}$ *(*$\mathbf{CFL_e}^-$*) consisting of focused axioms (context-free focused axioms), semi-analytic rules, and any subset of* $\mathcal{R}$*, then* $\mathbf{H}$ *has* $\mathbf{H}$*-interpolation.*

Combining Theorems 4.4, 4.6, and 4.10 we get the main corollary of this section:

Corollary 4.11. *Let* $\mathbf{G}$ *be a sequent calculus for the logic* L *with the condition that if* $\mathbf{G}$ *contains a subset of* $\mathcal{R}$*, then it satisfies the modal admissibility conditions. Then, if any of the following happens, the logic* L *has Craig interpolation.*

- $\mathsf{FL_e} \subseteq \mathsf{L}$*, (*$\mathsf{FL_e}^- \subseteq \mathsf{L}$*) and* $\mathbf{G}$ *is single-conclusion and consists of focused axioms (context-free focused axioms), semi-analytic rules, and any subset of* $\mathcal{R}$*;*

- $\mathsf{IPC} \subseteq \mathsf{L}$ *and* $\mathbf{G}$ *is single-conclusion and consists of focused axioms, semi-analytic rules, context-sharing semi-analytic rules, and any subset of* $\mathcal{R}$*; or*

- $\mathsf{CFL_e} \subseteq \mathsf{L}$*, (*$\mathsf{CFL_e}^- \subseteq \mathsf{L}$*) and* $\mathbf{G}$ *is multi-conclusion and consists of focused axioms (context-free focused axioms), semi-analytic rules, and any subset of* $\mathcal{R}$*.*

In the following, we present some applications of Corollary 4.11. Let us first consider a positive application:

Corollary 4.12. *The logics* $\mathsf{FL_e}$*,* $\mathsf{FL_{ec}}$*,* $\mathsf{FL_{ew}}$*,* $\mathsf{CFL_e}$*,* $\mathsf{CFL_{ew}}$*,* $\mathsf{CFL_{ec}}$*,* ILL*,* CLL*,* IPC*,* CPC *and their* K*,* KD *and* $\mathsf{S4}$ *versions have the Craig interpolation property. The same also goes for* $\mathsf{K4}$ *and* $\mathsf{KD4}$ *extensions of* IPC *and* CPC*.*

Proof. The cut-free sequent calculi for these logics presented in Preliminaries consist of focused axioms and semi-analytic rules. Therefore, by the Corollary 4.11 we can prove the Craig interpolation property for all of them. □

For the negative applications, we use the results in [5], [10] and [12] to ensure that the following logics do not enjoy the Craig interpolation property. Then, we will use Corollary 4.11 to prove that these logics do not have a semi-analytic calculus consisting only of focused axioms and semi-analytic rules.

Corollary 4.13. *The following logics do* ***not*** *contain a single- or multi-conclusion sequent calculus consisting only of focused axioms, semi-analytic rules, and rules from the set* $\mathcal{R}$*.*

- *Logics* R, UL^-, IUL^-, MTL, SMTL, IMTL, BL, L_∞, L_n *for* $n \geqslant 3$, P, CHL *and* A*;*

- *none of the consistent* BL*-extensions, except for* G, $\mathsf{G3}$ *and* CPC*;*

- *none of the consistent* IMTL*-extensions, except for* CPC*;*

- *none of the consistent extensions of* $\mathsf{RM^e}$*, except for* $\mathsf{RM^e}$, IUML^-, CPC, $\mathsf{RM^e_3}$, $\mathsf{RM^e_4}$, $\mathsf{CPC} \cap \mathsf{IUML}^-$, $\mathsf{RM^e_4} \cap \mathsf{IUML}^-$, *and* $\mathsf{CPC} \cap \mathsf{RM^e_3}$. *This category includes:*

  (*i*) $\mathsf{RM^e_n}$ *for* $n \geqslant 5$,

  (*ii*) $\mathsf{RM^e_{2m}} \cap \mathsf{RM^e_{2n+1}}$ *for* $n \geqslant m \geqslant 1$ *with* $n \geqslant 2$.,

  (*iii*) $\mathsf{RM^e_{2m}} \cap \mathsf{IUML}^-$ *for* $m \geqslant 3$*;*

- *none of the consistent super-intuitionistic logics, except for* IPC, LC, KC, $\mathsf{Bd_2}$, Sm, GSc *and* CPC*;*

- *none of the consistent extensions of* $\mathsf{S4}$*, except for at most thirty seven of them.*

5 Uniform Interpolation

In this section we will generalize the investigations of [8] to also cover the substructural setting and semi-analytic rules. We will show that any extension of a sequent calculus by semi-analytic rules preserves uniform interpolation if the resulted system turns out to be terminating. Our method here is similar to the method used in [8].

As a first step, let us generalize the notion of uniform interpolation from logics to sequent calculi. The following definition offers three versions of such a generalization, each of which suitable for different forms of rules.

Definition 5.1. Let **G** and **H** be two sequent calculi. **G** has **H**-*uniform interpolation* if for any sequent S and T where $T^s = \varnothing$ and any atom p, there exist p-free formulas $I(S)$ and $J(T)$ such that $V(I(S)) \subseteq V(S^a \cup S^s)$ and $V(J(T)) \subseteq V(T^a)$ and

(*i*) $S \cdot (I(S) \Rightarrow)$ is derivable in **H**.

(*ii*) For any p-free multiset Γ, if $S \cdot (\Gamma \Rightarrow)$ is derivable in $\mathbf{G}$ then $\Gamma \Rightarrow I(S)$ is derivable in $\mathbf{H}$.

(*iii*) $T \cdot (\Rightarrow J(T))$ is derivable in $\mathbf{H}$.

(*iv*) For any p-free multisets Γ and Δ, if $T \cdot (\Gamma \Rightarrow \Delta)$ is derivable in $\mathbf{G}$ then $J(T), \Gamma \Rightarrow \Delta$ is derivable in $\mathbf{H}$.

Similarly, we say $\mathbf{G}$ has *weak* $\mathbf{H}$-*uniform interpolation* if instead of (*ii*) we have

(*ii*′) For any p-free multiset Γ, if $S \cdot (\Gamma \Rightarrow)$ is derivable in $\mathbf{G}$ then $J(\tilde{S}), \Gamma \Rightarrow I(S)$ is derivable in $\mathbf{H}$ where $\tilde{S} = (S^a \Rightarrow)$.

We say $\mathbf{G}$ has *strong* $\mathbf{H}$-*uniform interpolation* if instead of (*ii*) we have

(*ii*″) For any p-free multisets Γ and Δ, if $S \cdot (\Gamma \Rightarrow \Delta)$ is derivable in $\mathbf{G}$ then $\Gamma \Rightarrow I(S), \Delta$ is derivable in $\mathbf{H}$.

Note that in the case of the strong uniform interpolation, T^s can be non-empty, and we have multi-conclusion rules.

We call $I(S)$ a left p-interpolant (weak p-interpolant, strong p-interpolant) of S and $J(T)$ a right p-interpolant (weak right p-interpolant, strong right p-interpolant) of T in $\mathbf{G}$ relative to $\mathbf{H}$. The system $\mathbf{H}$ has unifrom interpolation property (weak unifrom interpolation property, strong unifrom interpolation property) if it has $\mathbf{H}$-uniform interpolation (weak $\mathbf{H}$-uniform interpolation, strong $\mathbf{H}$-uniform interpolation).

Theorem 5.2. *If* $\mathbf{G}$ *is a sequent calculus with (weak/strong) uniform interpolation and complete for a logic* L *extending (*$\mathbf{FL_e}$/$\mathbf{CFL_e}$*)* $\mathbf{FL_e}$*,* L *has the uniform interpolation property.*

Proof. First note that since $\mathbf{G}$ is complete for L, $\mathsf{L} \vdash \phi \to \psi$ iff $\mathbf{G} \vdash \phi \Rightarrow \psi$. Hence we can rewrite the definition of the uniform interpolation using the sequent system $\mathbf{G}$. Now pick $S = (\Rightarrow A)$. By uniform interpolation property of $\mathbf{G}$, there is a p-free formula $I(S)$ such that $S \cdot (I(S) \Rightarrow)$ and for any p-free Σ if $S \cdot (\Sigma \Rightarrow)$, then $\Sigma \Rightarrow I(S)$. It is clear that $I(S)$ works as the p-pre-interpolant of A, because firstly $I(S) \Rightarrow A$ and secondly if $B \Rightarrow A$ then $B \Rightarrow I(S)$ for any p-free B. The same argument also works for the p-post-interpolant. In the case of weak uniform interpolation, first note that by definition if $T = (\Rightarrow)$ then $(\Rightarrow J(T))$. Secondly, note that since $\mathbf{G}$ is complete for L, the calculus should admit the cut rule by Theorem 4.4. Now we claim that $I(S)$ works again. The reason now is that if $B \Rightarrow A$ for a p-free B, then $J(\tilde{S}), B \Rightarrow I(S)$. Since $\tilde{S} = T$ and we have the cut rule, $B \Rightarrow I(S)$. The case for strong uniform interpolation is similar to the interpolation case. □

In the following theorem, we will check the uniform interpolation property for a set of focused axioms. It can also be considered as an example to show how this notion works in practice.

Theorem 5.3. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be two sequent calculi such that every provable sequent in* $\mathbf{G}$ *is also provable in* $\mathbf{H}$ *and* $\mathbf{G}$ *consists only of finite focused axioms. Then:*

- (i) *If* $\mathbf{G}$ *and* $\mathbf{H}$ *are single-conclusion and* $\mathbf{H}$ *extends* $\mathbf{FL_e}$*, then* $\mathbf{G}$ *has* $\mathbf{H}$*-uniform interpolation.*
- (ii) *If* $\mathbf{G}$ *and* $\mathbf{H}$ *are single-conclusion and* $\mathbf{H}$ *extends* $\mathbf{FL_e}$ *and has the left weakening rule, then* $\mathbf{G}$ *has weak* $\mathbf{H}$*-uniform interpolation.*
- (iii) *If* $\mathbf{G}$ *and* $\mathbf{H}$ *are multi-conclusion and* $\mathbf{H}$ *extends* $\mathbf{CFL_e}$*, then* $\mathbf{G}$ *has strong* $\mathbf{H}$*-uniform interpolation.*

Proof. To prove part (i) of the theorem, we have to find $I(S)$ and $J(T)$ for given sequents $S = (\Sigma \Rightarrow \Lambda)$ and $T = (\Pi \Rightarrow)$ such that the four conditions in the Definition 5.1 hold. We will denote our $I(S)$ and $J(T)$ by $\forall pS$ and $\exists pT$, respectively.

First, we will prove (i) and we will investigate the case $\exists pT$, first. For that purpose, define $\exists pT$ as the following

$$[(\circledast \Pi_p) * \top] \wedge 0 \wedge \bot$$

where Π_p is the subset of Π consisting of all p-free formulas and by $\circledast \Pi_p$ we mean $\phi_1 * \cdots * \phi_k$, where $\{\phi_1, \cdots, \phi_k\} = \Pi_p$. Note that $\top$ appears in the first conjunct only when $\Pi - \Pi_p$ is non-empty. Moreover, 0 only appears as a conjunct when T is of the form axiom 3 (which is $\bar{\beta} \Rightarrow$) and $\bar{\beta} = \Pi$, and $\bot$ only appears as a conjunction when T is of the form of axiom 4 (which is $\Sigma, \bar{\phi} \Rightarrow \Lambda$) and we have $\bar{\phi} \subseteq \Pi$.

First, we have to show that $\Pi \Rightarrow \exists pT$ holds in $\mathbf{H}$. Note that Π is of the form $\Pi_p \cup (\Pi - \Pi_p)$. By definition, for every $\psi \in \Pi_p$ we have $\psi \Rightarrow \psi$ and hence using the rule $(R*)$ we have $\Pi_p \Rightarrow \circledast \Pi_p$ holds in $\mathbf{H}$ (note that since $\mathbf{H}$ extends $\mathbf{FL_e}$, it has the rule $(R*)$). On the other hand, using the axiom for $\top$ we have $\Pi - \Pi_p \Rightarrow \top$ and then using the rule $(R*)$ we have $\Pi_p, \Pi - \Pi_p \Rightarrow (\circledast \Pi_p) * \top$, which is $\Pi \Rightarrow (\circledast \Pi_p) * \top$.

The formula 0 appears as a conjunct when T is of the form axiom 3 and $\bar{\beta} = \Pi$, which means that in this case $\Pi \Rightarrow$ is an instance of axiom 3 and it holds in $\mathbf{H}$. Hence, using the rule $(0w)$ we have $\Pi \Rightarrow 0$.

The formula $\bot$ appears as a conjunct when T is of the form axiom 4 and $\bar{\phi} \subseteq \Pi$. Hence, $\Pi \Rightarrow \bot$ is an instance of axiom 4 when we let Δ to be $\bot$.

Now, we have to show that if for p-free sequents $\bar{C}$ and $\bar{D}$ if $\Pi, \bar{C} \Rightarrow \bar{D}$ is provable in $\mathbf{G}$, then $\exists pT, \bar{C} \Rightarrow \bar{D}$ is provable in $\mathbf{H}$. Therefore, $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of one of the focused axioms and we have five cases to consider:

(1) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\phi \Rightarrow \phi$. Then, since $\bar{D} = \phi$, it means that ϕ is p-free. There are two cases; first, if $\Pi = \phi$ and $\bar{C} = \varnothing$, then $\circledast \Pi_p = \phi$ and since $\Pi - \Pi_p = \varnothing$, we do not have $\top$ in the conjunct. Hence, $\Pi \Rightarrow \phi$ and using the rule $(L\wedge)$ we have $\exists pT \Rightarrow \bar{D}$. Second, if $\Pi = \varnothing$ and $\bar{C} = \phi$, then $\circledast \Pi_p = 1$ and since $\Pi - \Pi_p = \varnothing$, then $\top$ does not appear in the first conjunct in the definition of $\exists pT$. Hence, since $\bar{C} \Rightarrow \bar{D}$ is equal to $\phi \Rightarrow \phi$ and this is of the form of the axiom 1, using the rule $(1w)$ we have $1, \phi \Rightarrow \phi$ and using $(L\wedge)$ we have $\exists pT, \bar{C} \Rightarrow \bar{D}$.

(2) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\Rightarrow \bar{\alpha}$. Then, since $\bar{D} = \bar{\alpha}$, it means that $\bar{\alpha}$ is p-free and $\Pi = \bar{C} = \varnothing$. Hence, like the above case $\circledast \Pi_p = 1$ and we do not have $\top$ in the definition, either. Again, using the rule $(1w)$ we have $1 \Rightarrow \bar{\alpha}$ and by $(L\wedge)$ we have $\exists pT \Rightarrow \bar{\alpha}$.

(3) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $(\bar{\beta} \Rightarrow)$. Then there are two cases; first if $\bar{\beta} = \Pi$, then we must have 0 as one of the conjuncts in the definition of $\exists pT$. We have $\bar{C} = \bar{D} = \varnothing$ and $0 \Rightarrow$ is an axiom in $\mathbf{H}$ and using the rule $(L\wedge)$ we have $\exists pT \Rightarrow$. Second, if $\Pi \subsetneq \bar{\beta}$, since we have $\bar{\beta} = \Pi, \bar{C}$ and $\bar{C}$ is p-free, and we have this condition that for any two formulas in $\bar{\beta}$ they have the same variables, we have Π is p-free,

as well, which means every formula in Π is p-free and $\Pi = \Pi_p$ and $\top$ does not appear in the definition of $\exists pT$. Hence, using the rule $(L*)$ on $\Pi, \bar{C} \Rightarrow$, we have $\circledast \Pi_p, \bar{C} \Rightarrow$ and by the rule $(L\wedge)$ we have $\exists pT, \bar{C} \Rightarrow$.

(4) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\Gamma, \bar{\phi} \Rightarrow \Delta$, then there are two cases; first if $\bar{\phi} \subseteq \Pi$, then by definition of $\exists pT$, $\bot$ is one of the conjuncts. Therefore, since $\bot, \bar{C} \Rightarrow \bar{D}$ is an instance of an axiom in **H**, using the rule $(L\wedge)$ we have $\exists pT, \bar{C} \Rightarrow \bar{D}$ is derivable in **H**. Second, if $\bar{\phi} \nsubseteq \Pi$, then at least one of the elements in $\bar{\phi}$ is in $\bar{C}$ and hence it is p-free. Therefore, by the condition that for any two formulas in $\bar{\phi}$ they have the same variables, $\bar{\phi}$ is p-free. Hence, there cannot be any element of $\bar{\phi}$ present in $\Pi - \Pi_p$ and hence $\bar{\phi} \subseteq \Pi_p, \bar{C}$ and then $\bar{\phi} \subseteq \Pi_p, \bar{C}, \top$. Therefore, we have $\Pi_p, \bar{C} \Rightarrow \bar{D}$ because it is of the form of the axiom $\Gamma, \bar{\phi} \Rightarrow \Delta$ of **G** and hence it is provable in **H**. Therefore, using the axiom $(L*)$ we have $(\circledast \Pi_p) * \top, \bar{C} \Rightarrow \bar{D}$ and by $(L\wedge)$, $\exists pT, \bar{C} \Rightarrow \bar{D}$. (Note that it is possible that $\Pi - \Pi_p$ is empty. It is easy to show that in this case the claim also holds. It is enough to drop $\top$ in the last part of the proof.)

(5) Consider the case where $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\Gamma \Rightarrow \bar{\phi}, \Delta$. Then, since $\bar{\phi} \subseteq \bar{D}$, we have $\exists pT, \bar{C} \Rightarrow \bar{D}$ is an instance of the same axiom $\Gamma \Rightarrow \bar{\phi}, \Delta$ when we substitute Γ by $\exists pT, \bar{C}$.

Now, we will investigate the case $\forall pS$ for S of the form $\Sigma \Rightarrow \Lambda$. Define $\forall pS$ as the following

$$[(\circledast \Sigma_p \to \bot)] \vee [\circledast(\bar{\beta} - \Sigma)] \vee \phi \vee 1 \vee \top$$

where in the first disjunct, Σ_p means the p-free part of Σ, the second disjunct appears whenever there exists an instance of an axiom of the form (3) in **G** where $\Sigma \subseteq \bar{\beta}$, $\Lambda = \varnothing$ and $\bar{\beta}$ is p-free. The third disjunct appears if $\Sigma = \varnothing$ and $\Lambda = \phi$ where ϕ is p-free. The fourth disjunct appears if $\Sigma \Rightarrow \Lambda$ equals to one of the instances of the axiom (1), (2), or (3) in **G**. And finally, the fifth disjunct appears when $\bar{\phi} \subseteq \Sigma$ for an instance of $\bar{\phi}$ in axiom (4) in **G** or $\bar{\phi} \subseteq \Lambda$ for an instance of $\bar{\phi}$ in axiom (5) in **G**.

First we have to show that $\Sigma, \forall pS \Rightarrow \Lambda$. For this purpose, we have to prove that for any possible disjunct X, we have $\Sigma, X \Rightarrow \Lambda$. For the first disjunct note that $\Sigma_p \Rightarrow \circledast \Sigma_p$ and $\Sigma - \Sigma_p, \bot \Rightarrow \Lambda$. Hence, $\Sigma, (\circledast \Sigma_p \to \bot) \Rightarrow$

Λ using the rule $(\to L)$.
For the second disjucnt, we have $\Sigma \subseteq \bar{\beta}$ and $\Lambda = \varnothing$. Therefore

$$\Sigma, *(\bar{\beta} - \Sigma) \Rightarrow \Lambda$$

by the axiom (3) itself. For the third disjunct, note that $\Sigma = \varnothing$ and $\Lambda = \phi$ where ϕ is p-free. Hence $\Sigma, \phi \Rightarrow \Lambda$ by axiom (1). For the fourth disjunct, note that $\Sigma \Rightarrow \Lambda$ is an axiom itself and hence $\Sigma, 1 \Rightarrow \Lambda$. Finally, for the fifth disjunct, note that $\Sigma \Rightarrow \Lambda$ is an instance of the axioms (4) or (5) which means if we also add $\top$ to the left hand-side of the sequent, it remains provable.

Now we have to prove that if $\Sigma, \bar{C} \Rightarrow \Lambda$ in **G** then $\bar{C} \Rightarrow \forall pS$ in **H**. For this purpose, we will check all possible axiomatic forms for $\Sigma, \bar{C} \Rightarrow \Lambda$.

(1) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (1), there are two possible cases. First if $\Sigma = \varnothing$ and $\bar{C} = \phi$ and $\Lambda = \phi$. Then ϕ will be p-free and hence ϕ appears in $\forall pS$ as a disjunct. Since $\bar{C} \Rightarrow \phi$, we have $\bar{C} \Rightarrow \forall pS$. For the second case, if $\Sigma = \phi$ and $\bar{C} = \varnothing$ then $\Sigma \Rightarrow \Lambda$ is an instance of the axiom (1) which means that 1 is a disjunct in $\forall pS$. Since $(\Rightarrow 1)$ and $\bar{C} = \varnothing$ we have $\bar{C} \Rightarrow \forall pS$.

(2) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (2). Then $\Sigma = \bar{C} = \varnothing$ and $\Lambda = \bar{\alpha}$. Therefore, 1 is a disjunct in $\forall pS$ and since $\bar{C} = \varnothing$ we have $\bar{C} \Rightarrow \forall pS$.

(3) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (3). Then there are two cases to consider. First if $\Sigma = \bar{\beta}$. Then $\bar{C} = \varnothing$ and $\Lambda = \varnothing$. By definition, 1 is a disjunct in $\forall pS$ and again like the previous cases $\bar{C} \Rightarrow \forall pS$. Second if $\Sigma \subsetneq \bar{\beta}$. Then $\bar{\beta} \cap \bar{C}$ is non-empty. Pick $\psi \in \bar{\beta} \cap \bar{C}$. ψ is p-free, since any pair of the elements in $\bar{\beta}$ have the same variables, $\bar{\beta}$ is p-free. Now by definition, $*(\bar{\beta} - \Sigma)$ is a disjunct in $\forall pS$. Since $\bar{C} = \beta - \Sigma$, we have $\bar{C} \Rightarrow \forall pS$.

(4) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (4). Similar to the previous case, there are two cases. If $\bar{\phi} \subseteq \Sigma$, then by definition $\top$ is a disjunct in $\forall pS$ and there is nothing to prove. In the second case, at least one the elements of ϕ is in $\bar{C}$ and hence p-free. Since any pair of the elements in $\bar{\phi}$ have the same variables, $\bar{\phi}$ is p-free. We can partition $\Sigma, \bar{C}$ to $\Sigma_p, \bar{C}, (\Sigma - \Sigma_p)$. Since every element of $(\Sigma - \Sigma_p)$ has p, and $\bar{\phi}$ is p-free,

the whole ϕ should belong to $\Sigma_p, \bar{C}$. Therefore, by the axiom (4) itself, $\Sigma_p, \bar{C} \Rightarrow \bot$ which implies $\bar{C} \Rightarrow (\circledast \Sigma_p \to \bot)$. By definition $(\circledast \Sigma_p) \to \bot$ is a disjunct in $\forall pS$ and hence $\bar{C} \Rightarrow \forall pS$.

(5) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (5). Then $\bar{\phi} \subseteq \Lambda$. By definition $\top$ is a disjunct in $\forall pS$ and therefore, there is nothing to prove.

For (ii), note that using the part (i) we have formulas $\exists pT$ and $\forall pS$ for any sequents S and T ($T^s = \varnothing$) with the conditions of **H**-uniform interpolation. The conditions for the weak **H**-uniform interpolation is the same except for the second part of the left weak p-interpolant which demands that if $\Sigma, \bar{C} \Rightarrow \Lambda$, then $\exists p\tilde{S}, \bar{C} \Rightarrow \forall pS$. If we use the same uniform interpolants, we satisfy all the conditions of weak **H**-uniform interpolation. The reason is that except the mentioned condition, all of the others are the same as the conditions for **H**-interpolation and for the other condition, we can argue as follows: By $\Sigma, \bar{C} \Rightarrow \Lambda$, we have $\bar{C} \Rightarrow \forall pS$ and by the left weakening rule we will have $\exists p\tilde{S}, \bar{C} \Rightarrow \forall pS$.

For (iii), first note that proving the existence of the right interpolants is enough. It is sufficient to define $\forall pS = \neg \exists pS$ and using the assumption that $\mathbf{CFL_e}$ is admissible in **H** to reduce the conditions of $\forall pS$ to $\exists pS$. Now define $\exists pS$ for any $S = (\Sigma \Rightarrow \Lambda)$ as:

$$[(\circledast \Sigma_p) * \top] \wedge [\neg(\bot + (\textstyle\bigplus \Lambda_p))] \wedge 0 \wedge \bot$$

where by $\circledast \Sigma_p$ we mean $\psi_1 * \cdots * \psi_r$, where $\{\psi_1, \cdots, \psi_r\} = \Sigma_p$ and $\bigplus \Lambda_p$ is defined similarly. Note that in $[(\circledast \Sigma_p) * \top]$ the formula $\top$ appears iff $\Sigma \neq \Sigma_p$, and $\bot$ appears in the second conjunct iff $\Lambda \neq \Lambda_p$. The third conjunct appears if $\Sigma \Rightarrow \Lambda$ is an instance of an axiom of the forms (1), (2) and (3) in **G** and the fourth conjunct appears if $\Sigma \Rightarrow \Lambda$ is an instance of an axiom of the forms (4), (5) in **G**.

First, we have to show that $\Sigma \Rightarrow \exists pS, \Lambda$. For that purpose, we have to check that for any conjunct X we have $\Sigma \Rightarrow X, \Lambda$. For the first conjunct, if $\Sigma \neq \Sigma_p$ then note that $\Sigma_p \Rightarrow \circledast \Sigma_p$ and $\Sigma - \Sigma_p \Rightarrow \top, \Lambda$ therefore

$$\Sigma \Rightarrow [(\circledast \Sigma_p) * \top], \Lambda$$

If $\Sigma = \Sigma_p$, then there is no need for $\top$ and the claim is clear by $\Sigma \Rightarrow \circledast \Sigma_p$. For the second conjunct, if $\Lambda \neq \Lambda_p$ note that $\bigplus \Lambda_p \Rightarrow \Lambda_p$ and $\Sigma, \bot \Rightarrow \Lambda - \Lambda_p$, hence

$$\Sigma, [\bot + (\bigplus \Lambda_p)] \Rightarrow \Lambda$$

hence

$$\Sigma \Rightarrow [\neg(\bot + (\bigplus \Lambda_p))], \Lambda$$

If $\Lambda = \Lambda_p$, similar to the case before, there is no need for $\bot$.

The cases for the third and the fourth conjuncts are similar to the similar cases in the proof of (i).

Now we want to prove that if $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ in **G**, then $\exists pS, \bar{C} \Rightarrow \bar{D}$ in **H**. For this purpose, we will check all the cases one by one:

(1) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (1), we have four cases to check.

- If $\phi \in \bar{C}$ and $\phi \in \bar{D}$, then $\Sigma = \Lambda = \varnothing$ and $\bar{C} = \bar{D} = \phi$. Hence $\circledast \Sigma_p = 1$. Therefore, since $1, \bar{C} \Rightarrow \bar{D}$ we have $\exists pS, \bar{C} \Rightarrow \bar{D}$.
- If $\phi \in \bar{C}$ and $\phi \notin \bar{D}$ then $\Sigma = \varnothing$ and $\Lambda = \phi$. Therefore, ϕ is p-free and hence $\Lambda_p = \phi$. Since $\bar{D} = \varnothing$ and $\Lambda = \phi$, we have $, \neg\phi, \bar{C} \Rightarrow \bar{D}$. Therefore, $\neg(\bigplus \Lambda_p), \bar{C} \Rightarrow \bar{D}$.
- If $\phi \notin \bar{C}$ and $\phi \in \bar{D}$. This case is similar to the previous case.
- If $\phi \notin \bar{C}$ and $\phi \notin \bar{D}$ then $\Sigma = \Lambda = \phi$ and $\bar{C} = \bar{D} = \varnothing$. Hence, by definition, we have 0 as a conjunct in $\exists pS$. Since $0 \Rightarrow$, we will have $\exists pS, \bar{C} \Rightarrow \bar{D}$.

(2) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (2). Then $\Sigma = \bar{C} = \varnothing$. There are two cases to consider. If $\Lambda = \bar{\alpha}$. Then by definition 0 appears in $\exists pS$. Since $\bar{D} = \varnothing$ and $(0 \Rightarrow)$ we have $\bar{C}, \exists pS \Rightarrow \bar{D}$. If $\Lambda \subsetneq \bar{\alpha}$, then $\bar{D} \cap \bar{\alpha}$ is non empty. Therefore, there exists a p-free formula in $\bar{\alpha}$. Since the variables of any pair in $\bar{\alpha}$ are equal, $\bar{\alpha}$ is p-free. Therefore, $\Lambda \subseteq \bar{\alpha}$ is p-free, hence $\Lambda = \Lambda_p$ (and $\bot$ does not appear in the second conjunct). Since $(\Rightarrow \Lambda, \bar{D})$, we have $(\Rightarrow \bigplus \Lambda, \bar{D})$ therefore $(\neg(\bigplus \Lambda_p) \Rightarrow \bar{D})$ which implies $(\exists pS \Rightarrow \bar{D})$.

(3) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (3). This case is similar to the previous case (2).

(4) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (4). There are two cases to consider. If $\bar{\phi} \subseteq \Sigma$. Then by definition $\bot$ is a conjunct in $\exists pS$ and therefore there is nothing to prove. For the second case, if $\bar{\phi} \nsubseteq \Sigma$, then $\bar{\phi} \cap \bar{C}$ is non-empty. Hence, $\bar{\phi}$ has a p-free element. Since the variables of any pair in $\bar{\phi}$ are equal, $\bar{\phi}$ is p-free. Since $\bar{\phi} \subseteq \Sigma_p, \bar{C}, \Sigma - \Sigma_p$ and $\bar{\phi}$ is p-free, we should have $\bar{\phi} \subseteq \Sigma_p, \bar{C}$. Therefore, if $\Sigma \neq \Sigma_p$, by the axiom (4) itself, $\top, \Sigma_p, \bar{C} \Rightarrow \bar{D}$. Since $(\circledast \Sigma_p) * \top$ is a conjunct in $\exists pS$, we will have $\exists pS, \bar{C} \Rightarrow \bar{D}$. Note that if $\Sigma = \Sigma_p$, then we will use $\Sigma_p, \bar{C} \Rightarrow \bar{D}$ instead of $\top, \Sigma_p, \bar{C} \Rightarrow \bar{D}$.

(5) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (5). This case is similar to the previous case 4.

□

5.1 The Single-conclusion Case

In this section, we assume that for any sequent $S = \Gamma \Rightarrow \Delta$, the nimber of elements of Δ is at most one. We will show how the single-conclusion semi-analytic and context-sharing semi-analytic rules preserve the uniform interpolation property. For this purpose, we will investigate these two kinds of rules separately. First we will study the semi-analytic rules and then we will show in the presence of weakening and context-sharing implication rules, we can also handle the context-sharing semi-analytic rules.

5.1.1 Semi-analytic Case

Let us begin right away with the following theorem which is one of the main theorems of this paper.

Theorem 5.4. *Let* **G** *and* **H** *be two single-conclusion sequent calculi and* **H** *extends* $\mathbf{FL_e}$. *If* **H** *is a terminating sequent calculus axiomatically extending* **G** *with only single-conclusion semi-analytic rules, then if* **G** *has* **H***-uniform interpolation property, then so does* **H**.

Proof. For any sequent U and V where $V^s = \varnothing$ and any atom p, we define two p-free formulas, denoted by $\forall pU$ and $\exists pV$ and we will prove that they meet the conditions for the left and the right p-interpolants of U and V, respectively. We define them simultaneously and the definition uses recursion on the rank of sequents which is specified by the terminating condition of the sequent calculus **H**.

If V is the empty sequent we define $\exists pV$ as 1 and otherwise, we define $\exists pV$ as the following

$$(\bigwedge_{par} \mathop{\ast}_{i} \exists pS_i) \wedge (\bigwedge_{L\mathcal{R}} [(\mathop{\ast}_{j} \bigwedge_{s} \forall pT_{js}) * (\mathop{\ast}_{i \neq 1} \bigwedge_{r} \forall pS_{ir}) \to \bigvee_{r} \exists pS_{1r}]) \wedge (\Box \exists pV') \wedge (\exists^G pV).$$

In the first conjunct, the conjunction is over all non-trivial partitions of $V = S_1 \cdot \cdots \cdot S_n$ and i ranges over the number of S_i's, in this case $1 \leqslant i \leqslant n$. In the second conjunct, the first big conjunction is over all left semi-analytic rules that are backward applicable to V in **H**. Since **H** is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{js}\rangle_s\rangle_j$, $\langle\langle S_{ir}\rangle_r\rangle_{i\neq 1}$ and $\langle S_{1r}\rangle$ and the conclusion is V, where $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$ which means that S_{ir}'s are those who have context in the right side of the sequents (Δ_i) in the premises of the left semi-analytic rule. (Note that picking the block $\langle S_{1r}\rangle$ is arbitrary and we include all conjuncts related to any choice of $\langle S_{1r}\rangle$.) The conjunct $\Box\exists pV'$ appears in the definition whenever V is of the form $(\Box\Gamma \Rightarrow)$ and we consider V' to be $(\Gamma \Rightarrow)$. And finally, since **G** has the **H**-uniform interpolation property, by definition there exists $J(V)$ as right p-interpolant of V. We choose one such $J(V)$ and denote it as $\exists^G pV$ and include it in the definition.

If U is the empty sequent define $\forall pU$ as 0. Otherwise, define $\forall pU$ as the following

$$(\bigvee_{par} (\mathop{\ast}_{i\neq 1} \exists pS_i \to \forall pS_1)) \vee (\bigvee_{L\mathcal{R}} [(\mathop{\ast}_{j} \bigwedge_{s} \forall pT_{js}) * (\mathop{\ast}_{i} \bigwedge_{r} \forall pS_{ir})])$$

$$\vee (\bigvee_{R\mathcal{R}} (\mathop{\ast}_{i} \bigwedge_{r} \forall pS_{ir})) \vee (\Box \forall pU') \vee (\forall^G pU).$$

In the first disjunct, the big disjunction is over all partitions of $U = S_1 \cdot \cdots \cdot S_n$ such that for each $i \neq 1$ we have $S_i^s = \varnothing$ and $S_1 \neq U$. (Note that in this case, if $S^s = \varnothing$ it may be possible that for one $i \neq 1$ we have $S_i = U$. Then the first disjunct of the definition must be $\exists pU \to \forall pS_1$ where $\forall pS_1 = 0$. But this does not make any problem, since the definition of $\exists pU$ is prior to the definition of $\forall pU$.) In the second disjunct, the big disjunction is over all left semi-analytic rules that are backward applicable to U in **H**. Since **H** is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{js}\rangle_s\rangle_j$ and $\langle\langle S_{ir}\rangle_r\rangle_i$ and the conclusion is U. In the third disjunct, the big disjunction is over all right semi-analytic rules backward applicable to U in **H**. The premise of the rule is $\langle\langle S_{ir}\rangle_r\rangle_i$ and the conclusion is U. The fourth disjunct is on all semi-analytic modal rules with the result U and the premise U'. And finally, since **G** has the **H**-uniform interpolation property, by definition there exists $I(U)$ as left p-interpolant of U. We choose one such $I(U)$ and denote it as $\forall^G pU$ and include it in the definition.

To prove the theorem we use induction on the order of the sequents and we prove both cases $\forall pU$ and $\exists pV$ simultaneously. First note that both $\forall pU$ and $\exists pV$ are p-free by construction and since in all the rules the variables in the premises also occurs in the consequence, we have $V(\forall pU) \subseteq V(U^a) \cup V(U^s)$ and $V(\exists pV) \subseteq V(V^a)$. Secondly, we have to show that:

(i) $V \cdot (\Rightarrow \exists pV)$ is derivable in **H**.

(ii) $U \cdot (\forall pU \Rightarrow)$ is derivable in **H**.

We show them using induction on the order of the sequents U and V. When proving (i), we assume that (i) holds for sequents whose succedents are empty and with order less than the order of V and (ii) holds for any sequent with order less than the order of V. We have the same condition for U when proving (ii).

To prove (i), note that if V is the empty sequent, then by definition $\exists pV = 1$ and hence (i) holds. For the rest, we have to show that $V \cdot (\Rightarrow X)$ is derivable in **H** for any X that is one of the conjuncts in the definition of $\exists pV$. Then, using the rule $(R\wedge)$ it follows that $V \cdot (\Rightarrow \exists pV)$. Since V is of the form $\Gamma \Rightarrow$, we have to show $\Gamma \Rightarrow X$ is derivable in **H**.

- In the case that the conjunct is $(\bigwedge_{par} \circledast_i \exists pS_i)$, we have to show that for any non-trivial partition $S_1 \cdot \cdots \cdot S_n$ of V we have $\Gamma \Rightarrow \circledast_i \exists pS_i$ is derivable in **H**. Since the order of each S_i is less than the order of V and $S_i^s = (\Gamma_i \Rightarrow)$ for $1 \leqslant i \leqslant n$ where $\bigcup_{i=1}^{n} \Gamma_i = \Gamma$, we can use the induction hypothesis and we have $\Gamma_i \Rightarrow \exists pS_i$. Using the right rule for $(*)$ we have $\Gamma_1, \cdots, \Gamma_n \Rightarrow \circledast_i \exists pS_i$ which is $\Gamma \Rightarrow \circledast_i \exists pS_i$.

- For the second conjunct in the definition of $\exists pV$, we have to check that for every left semi-analytic rule we have

$$V \cdot (\Rightarrow [(\circledast_j \bigwedge_s \forall p T_{js}) * (\circledast_{i \neq 1} \bigwedge_r \forall p S_{ir}) \to \bigvee_r \exists p S_{1r}]).$$

is derivable in **H**. Therefore, V is the conclusion of a left semi-analytic rule such that the premises are $\langle\langle T_{js}\rangle_s\rangle_j$, $\langle\langle S_{ir}\rangle_r\rangle_i$ and $\langle S_{1r}\rangle_r$ and hence the order of all of them are less than the order of V. We can easily see that the claim holds since by induction hypothesis we can add $\forall p T_{js}$ and $\forall p S_{ir}$ to the left side of the sequents T_{js} and S_{ir} for $i \neq 1$. And again by induction hypothesis we can add $\exists p S_{1r}$ to the right side of the sequents S_{1r}. Then using the rules $L\wedge$, $L*$ and $R\vee$ the claim follows. What we have said so far can be seen precisely in the following:

Note that $\langle\langle T_{js}\rangle_s\rangle_j$ is of the form $\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j$ and $\langle\langle S_{ir}\rangle_r\rangle_i$ is of the form $\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_i$ and V is of the form

$$\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow$$

Using induction hypothesis we have for every $1 \leqslant j \leqslant m$

$$(\Pi_j, \forall p T_{j1}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \cdots, (\Pi_j, \forall p T_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \cdots$$

for every $1 < i \leqslant n$ we have

$$(\Gamma_i, \forall p S_{i1}, \bar{\phi}_{i1} \Rightarrow), \cdots, (\Gamma_i, \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow), \cdots$$

and for $i = 1$ we have

$$(\Gamma_1, \bar{\phi}_{11} \Rightarrow \exists p S_{11}), \cdots, (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \exists p S_{1r}), \cdots$$

Hence, using the rule $(L\wedge)$, for every $1 \leqslant j \leqslant m$ we have

$$(\Pi_j, \bigwedge_s \forall p T_{js}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \cdots, (\Pi_j, \bigwedge_s \forall p T_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \cdots$$

and for every $1 < i \leqslant n$ we have

$$(\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{i1} \Rightarrow), \cdots (\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow), \cdots$$

and using the rule $(R\vee)$, for $i = 1$ we have

$$(\Gamma_1, \bar{\phi}_{11} \Rightarrow \bigvee_r \exists p S_{1r}), \cdots, (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r \exists p S_{1r}) \cdots$$

Substituting all these three in the original left semi-analytic rule (we can do this, since in the original rule, there are contexts, Δ_i's in the right hand side of the sequents $S'_{ir}s$), we conclude

$$\Pi, \Gamma, \phi, \langle \bigwedge_s \forall p T_{js} \rangle_j, \langle \bigwedge_r \forall p S_{ir} \rangle_{i \neq 1} \Rightarrow \bigvee_r \exists p S_{1r}.$$

where $\Pi = \Pi_1, \cdots, \Pi_m$, $\Gamma = \Gamma_1, \cdots, \Gamma_n$, $\langle \bigwedge_s \forall p T_{js} \rangle_j = \bigwedge_s \forall p T_{1s}, \cdots, \bigwedge_s \forall p T_{ms}$ and $\langle \bigwedge_r \forall p S_{ir} \rangle_{i \neq 1} = \bigwedge_r \forall p S_{2r}, \cdots, \bigwedge_r \forall p S_{nr}$.
Now, using the rule $(L*)$ we have

$$\Pi, \Gamma, \phi, (\mathop{\circledast}_j \bigwedge_s \forall p T_{js}) * (\mathop{\circledast}_{i \neq 1} \bigwedge_r \forall p S_{ir}) \Rightarrow \bigvee_r \exists p S_{1r}.$$

And finally, using the rule $R \to$ we conclude

$$\Pi, \Gamma, \phi \Rightarrow [(\mathop{\circledast}_j \bigwedge_s \forall p T_{js}) * (\mathop{\circledast}_{i \neq 1} \bigwedge_r \forall p S_{ir}) \to \bigvee_r \exists p S_{1r}].$$

- Consider the conjunct $\Box \exists p T'$. In this case, T must have been of the form $(\Box \Gamma \Rightarrow)$ and T' of the form $(\Gamma \Rightarrow)$. By definition, the order of T' is less than the order of T. Hence, by induction hypothesis we have $T' \cdot (\Rightarrow \exists p T')$ or in other words $\Gamma \Rightarrow \exists p T'$. Now, we use the rule K and we have $\Box \Gamma \Rightarrow \Box \exists p T'$ which means $T \cdot (\Rightarrow \Box \exists p T')$.

- The last case is $\exists^G p V$. We have to show $V \cdot (\Rightarrow \exists^G p V)$ is provable in **H** which is the case since **G** has **H**-uniform interpolation property and by Definition 5.1 part (iii) there exists p-free formula J such that $V \cdot (\Rightarrow J)$ is derivable in **H**. We chose one such J and call it $\exists^G p V$, hence $V \cdot (\Rightarrow \exists^G p V)$ in **H** by definition.

To prove (ii), note that if U is the empty sequent, then by definition $\forall p U = 0$ and hence (ii) holds. For the rest, we have to show that $U \cdot (X \Rightarrow)$ is derivable in **H** for any X that is one of the disjuncts in the definition of $\forall p U$. Then, using the rule $(L\vee)$ it follows that $U \cdot (\forall p U \Rightarrow)$. Since U is of the form $\Gamma \Rightarrow \Delta$, we have to show that $\Gamma, X \Rightarrow \Delta$ is derivable in **H**.

- In the case that the disjunct is $(\bigvee_{par}(\underset{i\neq 1}{\circledast} \exists p S_i \to \forall p S_1))$ we have to prove that for any partitions of $U = S_1 \cdot \cdots \cdot S_n$ such that $S_i^s = \varnothing$ for each $i \neq 1$ and $S_1 \neq U$, we have $U \cdot ((\underset{i\neq 1}{\circledast} \exists p S_i \to \forall p S_1) \Rightarrow)$. First, consider the case that none of S_i's are equal to U (or in other words, $S^s \neq \varnothing$); then the order of each S_i is less than the order of S and we can use the induction hypothesis. Since for $i \neq 1$ the succedent of each S_i is empty, we have $S_i = (\Gamma_i \Rightarrow)$ and $(\Gamma_i \Rightarrow \exists p S_i)$ and using the rule $R*$ we have $(\Gamma_2, \cdots, \Gamma_n \Rightarrow \underset{i\neq 1}{\circledast} \exists p S_i)$. And for $S_1 = \Gamma_1 \Rightarrow \Delta$ we have $\Gamma_1, \forall p S_1 \Rightarrow \Delta$. Hence using the rule $L\to$ we conclude

  $$\Gamma_1, \cdots, \Gamma_n, \underset{i\neq 1}{\circledast} \exists p S_i \to \forall p S_1 \Rightarrow \Delta$$

  and the claim follows.
  In the case that $U^s = \varnothing$, it is possible that for $i \neq 1$, one of S_i's is equal to U. In this case what appears in the definition of $\forall p U$ is $\exists p U \to \forall p S_1$ which is equivalent to $\exists p U \to 0$. But, we can do this, since we defined $\exists p U$ prior to the definition of $\forall p U$ and we have proved $U \cdot (\Rightarrow \exists p U)$ prior to the case that we are checking now.

- In the case that the disjunct is $(\bigvee_{L\mathcal{R}}[(\underset{j}{\circledast} \bigwedge_s \forall p T_{js}) * (\underset{i}{\circledast} \bigwedge_r \forall p S_{ir})])$, we have to prove that for any left semi-analytic rule that is backward applicable to U in **H** we have $U \cdot ((\underset{j}{\circledast} \bigwedge_s \forall p T_{js}) * (\underset{i}{\circledast} \bigwedge_r \forall p S_{ir}) \Rightarrow)$. The premises of the rule are $\langle\langle T_{js}\rangle_s\rangle_j$ and $\langle\langle S_{ir}\rangle_r\rangle_i$ and the conclusion is U. Since the orders of all T_{js}'s and S_{ir}'s are less than the order of U we can use the induction hypothesis and have $T_{js} \cdot (\forall p T_{js} \Rightarrow)$ and $S_{ir} \cdot (\forall p S_{ir} \Rightarrow)$. Using the rule $(L\wedge)$ for context sharing sequents (when j is fixed and i is fixed we have context sharing sequents) and then using the rule $(L*)$ for non context sharing sequents (when s and r are fixed and we are ranging over j and i) and then applying the same left rule we can prove the claim. The proof is similar to the second case of (i) and precisely it goes as the following: Using induction hypothesis we have for every $1 \leqslant j \leqslant m$

  $$(\Pi_j, \forall p T_{j1}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \cdots, (\Pi_j, \forall p T_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \cdots$$

  and for every $1 \leqslant i \leqslant n$ we have

  $$(\Gamma_i, \forall p S_{i1}, \bar{\phi}_{i1} \Rightarrow \Delta_i), \cdots, (\Gamma_i, \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow \Delta_i), \cdots$$

Hence, using the rule $(L\wedge)$, for every $1 \leqslant j \leqslant m$ we have

$$(\Pi_j, \bigwedge_s \forall p T_{js}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \cdots, (\Pi_j, \bigwedge_s \forall p T_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \cdots$$

and for every $1 \leqslant i \leqslant n$ we have

$$(\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{i1} \Rightarrow \Delta_i), \cdots, (\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow \Delta_i), \cdots$$

Substituting these two in the original left semi-analytic rule, we conclude

$$\Pi, \Gamma, \phi, \langle \bigwedge_s \forall p T_{js} \rangle_j, \langle \bigwedge_r \forall p S_{ir} \rangle_i \Rightarrow \Delta,$$

and using the rule $(L*)$ we have

$$\Pi, \Gamma, \phi, (\circledast_j \bigwedge_s \forall p T_{js}) * (\circledast_i \bigwedge_r \forall p S_{ir}) \Rightarrow \Delta.$$

- In the case that the disjunt is $(\bigvee_{R\mathcal{R}} (\circledast_i \bigwedge_r \forall p S_{ir}))$, we have to prove that for any right semi-analytic rule backward applicable to U in $\mathbf{H}$, we have $U \cdot (\circledast_i \bigwedge_r \forall p S_{ir} \Rightarrow)$. In this case the premises of the rule are $\langle\langle S_{ir} \rangle_r \rangle_i$, where $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir})$ and the conclusion is $U = (\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi)$. Since the order of each S_{ir} is less than the order of S, we can use the induction hypothesis and for every $1 \leqslant i \leqslant n$ we have

  $$(\Gamma_i, \forall p S_{i1}, \bar{\phi}_{i1} \Rightarrow \bar{\psi}_{i1}), \cdots, (\Gamma_i, \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}), \cdots$$

  Using the rule $L\wedge$ we have

  $$(\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{i1} \Rightarrow \bar{\psi}_{i1}), \cdots, (\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}), \cdots$$

  and substituting it in the original right rule, we conclude

  $$\Gamma, \langle \bigwedge_r \forall p S_{ir} \rangle_i \Rightarrow \phi,$$

  and using the rule $(L*)$ we have

  $$\Gamma, \circledast_i \bigwedge_r \forall p S_{ir} \Rightarrow \phi.$$

- ◦ For the case that the disjunct is $\Box\forall pU'$ we have that U is the conclusion of a semi-analytic modal rule and the premise is U'. Hence, U is of the form $(\Box\Gamma \Rightarrow \Box\Delta)$ and U' is of the form $(\Gamma \Rightarrow \Delta)$. Since the order of U' is less than the order of U, we can use the induction hypothesis and we have $(\Gamma, \forall pU' \Rightarrow \Delta)$. Now, using the rule K we can conclude $(\Box\Gamma, \Box\forall pU' \Rightarrow \Box\Delta)$ which is equivalent to $U \cdot (\Box\forall pU' \Rightarrow)$.

- ◦ And finally, for the case that the disjunct is $\forall^G pU$ we have to show that $U \cdot (\forall^G pU \Rightarrow)$ holds in **H**, which does since **G** has **H**-uniform interpolation property and by Definition 5.1 part (i) there exists p-free formula I such that $U \cdot (I \Rightarrow)$ is derivable in **H**. We choose one such I and call it $\forall^G pU$ and hence we have $U \cdot (\forall^G pU \Rightarrow)$ in **H** by definition.

So far we have proved (i) and (ii). We want to show that **H** has **H**-uniform interpolation. Therefore, based on the Definition 5.1, we have to prove the following, as well:

(iii) For any p-free multisets $\bar{C}$ and $\bar{D}$, if $V \cdot (\bar{C} \Rightarrow \bar{D})$ is derivable in **G** then $\exists pV, \bar{C} \Rightarrow \bar{D}$ is derivable in **H**, where $\bar{C} = C_1, \cdots, C_k$ and $|\bar{D}| \leqslant 1$.

(iv) For any p-free multiset $\bar{C}$, if $U \cdot (\bar{C} \Rightarrow)$ is derivable in **G** then $\bar{C} \Rightarrow \forall pU$ is derivable in **H**, where $\bar{C} = C_1, \cdots, C_k$.

Recall that V is of the form $(\Gamma \Rightarrow)$ and U is of the form $(\Gamma \Rightarrow \Delta)$. We will prove (iii) and (iv) simultaneously using induction on the length of the proof and induction on the order of U and V. More precisely, first by induction on the order of U and V and then inside it, by induction on n, we will show:

- For any p-free multisets $\bar{C}$ and $\bar{D}$, if $V \cdot (\bar{C} \Rightarrow \bar{D})$ has a proof in **G** with length less than or equal to n, then $\exists pV, \bar{C} \Rightarrow \bar{D}$ is derivable in **H**.

- For any p-free multiset $\bar{C}$, if $U \cdot (\bar{C} \Rightarrow)$ has a proof in **G** with length less than or equal to n, then $\bar{C} \Rightarrow \forall pU$ is derivable in **H**.

Where by the length we mean counting just the new rules that **H** adds to **G**.

First note that for the empty sequent and for (iii), we have to show that if $\bar{C} \Rightarrow \bar{D}$ is valid in **G**, then $\bar{C}, 1 \Rightarrow \bar{D}$ is valid in **H**, which is trivial by the rule $(1w)$. Similarly, for (iv), if $\bar{C} \Rightarrow$ is valid in **G**, then $\bar{C} \Rightarrow 0$ is valid in **H**, which is trivial by the rule $(0w)$.

For the base of the other induction, note that if $n = 0$, for (iii) it means that $\Gamma, \bar{C} \Rightarrow \bar{D}$ is valid in **G**. By Definition 5.1 part (iv), $\exists^G pV, \bar{C} \Rightarrow \bar{D}$ and hence $\exists pV, \bar{C} \Rightarrow \bar{D}$ is provable in **H**. For (iv), it means that $\Gamma, \bar{C} \Rightarrow \Delta$ is valid in **G**. Therefore, again by Definition 5.1, $\bar{C} \Rightarrow \forall^G pU$ and hence $\bar{C} \Rightarrow \forall pU$ is provable in **H**.

For $n \neq 0$, to prove (iii), we have to consider the following cases:

- The case that the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a left semi-analytic rule and $\phi \in \bar{C}$ (which means that the main formula of the rule, ϕ, is one of C_i's). Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$ is the conclusion of a left semi-analytic rule and V is of the form $(\Pi, \Gamma \Rightarrow)$ and $\bar{C} = (\bar{X}, \bar{Y}, \phi)$ and we want to prove $(\exists pV, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r\rangle_i}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$, $\bigcup_i \bar{Y}_i = \bar{Y}$ and $\bigcup_i \Delta_i = \Delta$. Consider $T_{js} = (\Pi_j \Rightarrow)$ and $S_{ir} = (\Gamma_i \Rightarrow)$. Since T_{js}'s do not depend on the suffix s, we have $T_{j1} = \cdots = T_{js}$ and we denote it by T_j. And, since S_{ir}'s do not depend on r, we have $S_{i1} = \cdots = S_{ir}$ and we denote it by S_i. Therefore, $T_1, \cdots, T_m, S_1, \cdots, S_n$ is a partition of V. First, consider the case that it is a non-trivial partition. Then the order of all of them are less than the order of V and since the rule is semi-analytic and ϕ is p-free then $\bar{\psi}_{js}$, $\bar{\theta}_{js}$ and $\bar{\phi}_{ir}$ are also p-free. Hence, we can use the induction hypothesis to get:

$$\exists pT_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \quad , \quad \exists pS_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i$$

If we let $\{\exists pT_j, \bar{X}_j\}$ and $\{\exists pS_i, \bar{Y}_i\}$ be the contexts in the original left semi-analytic rule, we have the following

$$\frac{\langle\langle \exists pT_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \rangle_s\rangle_j \qquad \langle\langle \exists pS_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i \rangle_r\rangle_i}{\exists pT_1, \cdots, \exists pT_m, \exists pS_1, \cdots, \exists pS_n, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

Using the rule $(L*)$ we have

$$(\circledast_j \exists pT_j) * (\circledast_i \exists pS_i), \bar{X}, \bar{Y}, \phi \Rightarrow \Delta.$$

Therefore using the rule $(L\wedge)$, we have $(\exists pV, \bar{C} \Rightarrow \bar{D})$.

If $T_1, \cdots, T_m, S_1, \cdots, S_n$ is a trivial partition of V, it means that one of them equals V and all the others are empty sequents. W.l.o.g. suppose $T_1 = V = (\Sigma \Rightarrow)$ and the others are empty. Then we must have had the following instance of the rule:

$$\frac{\langle\langle \Sigma, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i\rangle_r\rangle_i}{\Sigma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

Therefore, $V \cdot (\bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js})$ for every j and s are premises of $V \cdot (\bar{C} \Rightarrow \bar{D})$, and hence the length of their trees are smaller than the length of the proof tree of $V \cdot (\bar{C} \Rightarrow \bar{D})$, and since the rule is semi-analytic and ϕ is p-free then $\bar{\psi}_{js}$ and $\bar{\theta}_{js}$ are also p-free. Hence, for all of them we can use the induction hypothesis (induction on the length of the proof), and we have $\exists pV, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}$. Substituting $\{\exists pV, \bar{X}_j\}$, $\{\bar{X}_j\}$, $\{\bar{Y}_i\}$ and $\{\Delta\}$ as the contexts of the premises in the original left rule we have

$$\frac{\langle\langle \exists pV, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i\rangle_r\rangle_i}{\exists pV, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

which is $(\exists pV, \bar{C} \Rightarrow \bar{D})$.

- Consider the case where the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a left semi-analytic rule and $\phi \notin \bar{C}$. Therefore,

$$V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$$

is the conclusion of a left semi-analytic rule and V is of the form $(\Pi, \Gamma, \phi \Rightarrow)$ and $\bar{C} = (\bar{X}, \bar{Y})$ and we want to prove $(\exists pV, \bar{X}, \bar{Y} \Rightarrow \Delta)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma_1, \bar{Y}_1, \bar{\phi}_{1r} \Rightarrow \Delta\rangle_r}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta} \ (\dagger)$$

where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$ and $\bigcup_i \bar{Y}_i = \bar{Y}$.
Since, $\bar{X}_j$'s and $\bar{Y}_i$'s are in the context positions in the original rule, we

can consider the same substition of meta-sequents and meta-formulas as above in the original rule, except that we do not take $\bar{X}_j$'s and $\bar{Y}_i$'s as contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta\rangle_r}{\Pi, \Gamma, \phi \Rightarrow \Delta}$$

If we let $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow)$ for $i \neq 1$ and $S_{1r} = (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta)$, we can claim that this rule is back ward applicable to V and T_{js}'s and S_{ir}'s are the premises of the rule. Hence, their orders are less than the order of V and we can use the induction hypothesis for them. Note that we have $V \cdot (\bar{C} \Rightarrow \bar{D})$ is provable in **H** and from (†) we have that $T_{js} \cdot (\bar{X}_j \Rightarrow)$ and for $i \neq 1$, $S_{ir} \cdot (\bar{Y}_i \Rightarrow)$ and $S_{1r} \cdot (\bar{Y}_1 \Rightarrow \Delta)$ are also provable in **H**. Using the induction hypothesis we get

$$(\bar{X}_j \Rightarrow \forall p T_{js}) \quad , \quad (\bar{Y}_i \Rightarrow \forall p S_{ir})_{i\neq 1} \quad , \quad (\bar{Y}_1, \exists p S_{1r} \Rightarrow \Delta)$$

Note that we were allowed to use the induction hypothesis because for $i \neq 1$ we have $\Delta_i = \varnothing$ and Δ is p-free and T_{js}'s and S_{ir}'s meet the conditions of (iii) and (iv) in the induction step. Now, using the rules $(R\wedge)$ and $(L\vee)$ we have

$$(\bar{X}_j \Rightarrow \bigwedge_s \forall p T_{js}) \quad , \quad (\bar{Y}_i \Rightarrow \bigwedge_r \forall p S_{ir})_{i\neq 1} \quad , \quad (\bar{Y}_1, \bigvee_r \exists p S_{1r} \Rightarrow \Delta)$$

Denote $(\bigwedge_s \forall p T_{js})$ as A_j and $(\bigwedge_r \forall p S_{ir})$ as B_i (for $i \neq 1$) and $(\bigvee_r \exists p S_{1r})$ as C. We have

$$\dfrac{\dfrac{\dfrac{\langle \bar{X}_j \Rightarrow A_j\rangle_j}{\bar{X} \Rightarrow \circledast_j A_j}\,R* \qquad \dfrac{\langle \bar{Y}_i \Rightarrow B_i\rangle_{i\neq 1}}{Y_2, \cdots, Y_n \Rightarrow \circledast_{i\neq 1} B_i}\,R*}{\bar{X}, Y_2, \cdots, Y_n \Rightarrow (\circledast_j A_j) * (\circledast_{i\neq 1} B_i)}\,R* \qquad \bar{Y}_1, C \Rightarrow \Delta}{\bar{X}, \bar{Y}, (\circledast_j A_j) * (\circledast_{i\neq 1} B_i) \to C \Rightarrow \Delta}\,L\to$$

Note that $(\circledast_j A_j) * (\circledast_{i\neq 1} B_i) \to C$ is defined as the second conjunct in the definition of $\exists p V$ and hence using the rule $(L\wedge)$ we have $(\exists p V, \bar{C} \Rightarrow \Delta)$.

∘ Consider the case when the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a right semi-analytic rule. Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{C} \Rightarrow \phi)$ is the conclusion of a right semi-analytic rule and V is of the form $(\Gamma \Rightarrow)$ and $\bar{D} = \phi$ and we want to prove $(\exists p V, \bar{C} \Rightarrow \phi)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle \Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma, \bar{C} \Rightarrow \phi}$$

where $\bigcup_i \Gamma_i = \Gamma$ and $\bigcup_i \bar{C}_i = \bar{C}$. Denote $(\Gamma_i \Rightarrow)$ as S_i. Then we have that $S_1, \cdots, S_n$ is a partition of V. First consider the case where it is a non-trivial partition of V. Therefore, the order of any S_i is less than the order of V and since the rule is semi-analytic and ϕ is p-free then $\bar{\psi}_{ir}$ and $\bar{\phi}_{ir}$ are also p-free, we can use the induction hypothesis on the order, and get

$$\exists p S_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}$$

Now, substituting $\{\exists p S_i, \bar{C}_i\}$ as the context in the original rule, we get

$$\exists p S_1, \cdots, \exists p S_n, \bar{C}_1, \cdots, \bar{C}_n \Rightarrow \phi$$

then using the rule $(L*)$ we have

$$\underset{i}{\ast} \exists p S_i, \bar{C} \Rightarrow \phi$$

and since $\underset{i}{\ast} \exists p S_i$ appears as the first conjunct in the definition of $\exists p V$, using the rule $(L\wedge)$ we have $(\exists p V, \bar{C} \Rightarrow \phi)$.
It remains to investigate the case where $S_1, \cdots, S_n$ is a trivial partition of V. W.l.o.g. suppose $S_1 = V$ and all the others are the empty sequents. Hence, we must have had the following instance of the rule

$$\frac{\langle \Gamma, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r} \rangle_r \qquad \langle\langle\ \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_{i \neq 1}}{\Gamma, \bar{C} \Rightarrow \phi}$$

We have, for all r, $V \cdot (\bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r})$ are the premises of $V \cdot (\bar{C} \Rightarrow \phi)$. Hence the length of tree proofs of all of them are less than the length of proof of $V \cdot (\bar{C} \Rightarrow \phi)$ and since the rule is semi-analytic and ϕ

is p-free then $\bar{\psi}_{1r}$ and $\bar{\phi}_{1r}$ are also p-free, we can use the induction hypothesis (induction on the length of proof) and get $\exists pV, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r}$. Substituting $\{\exists pV, \bar{C}_1\}$ as the context in the original semi-analytic rule we get

$$\frac{\langle \exists pV, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r}\rangle_r \qquad \langle\langle\ \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_{i\neq 1}}{\exists pV, \bar{C} \Rightarrow \phi}$$

which is what we wanted.

- And the final case is when the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a semi-analytic modal rule. Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box\Delta})$ is the conclusion of a semi-analytic modal rule and V is of the form $(\Box\Gamma \Rightarrow)$ and $\bar{C} = \overline{\Box C'}$ and $\bar{D} = \overline{\Box\Delta})$, where $|\overline{\Box\Delta}| \leqslant 1$ and $V' = (\Gamma \Rightarrow)$. We want to prove $(\exists pV, \bar{C} \Rightarrow \bar{D})$. We must have had the following instance of the rule

$$\frac{\Gamma, \bar{C'} \Rightarrow \bar{\Delta}}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box\Delta}}$$

Since the order of V' is less than the order of V, and C' and Δ are p-free, we can use the induction hypothesis and get

$$\exists pV', \bar{C'} \Rightarrow \bar{\Delta}$$

Using the rule K or D (depending on the cardinality of $\overline{\Box\Delta}$) we have $\Box\exists pV', \overline{\Box C'} \Rightarrow \overline{\Box\Delta}$ and since we have $\Box\exists pV'$ as one of the conjuncts in the definition of $\exists pV$, we conclude $\exists pV, \bar{C} \Rightarrow \bar{D}$ using the rule $(L\wedge)$.

Now, we have to prove (iv). Similar to the proof of part (iii), there are several cases to consider.

- Consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a left semi-analytic rule and $\phi \in \bar{C}$. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$ is the conclusion of a left semi-analytic rule and U is of the form $\Pi, \Gamma \Rightarrow \Delta$ and $\bar{C} = \bar{X}, \bar{Y}, \phi$ and we want to prove $\bar{X}, \bar{Y}, \phi \Rightarrow \forall pU$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$, $\bigcup_i \bar{Y}_i = \bar{Y}$ and $\bigcup_i \Delta_i = \Delta$. Consider $T_{js} = (\Pi_j \Rightarrow)$, $S_{1r} = \Gamma_1 \Rightarrow \Delta_1$, and for $i \neq 1$ let $S_{ir} = (\Gamma_i \Rightarrow)$. Since T_{js}'s do not depend on the suffix s, we have $T_{j1} = \cdots = T_{js}$ and we denote it by T_j. And, since S_{ir}'s do not depend on r for $i \neq 1$, we have $S_{21} = \cdots = S_{ir}$ and we denote it by S_i and with the same line of reasoning we denote S_{1r} by S_1. Therefore, $T_1, \cdots, T_m, S_1, \cdots, S_n$ is a partition of U. First, consider the case that S_1 does not equal U. Then the order of all of them are less than the order of U (or in some cases that the others can be equal to U, the length of their proof in the premises is lower) and since the rule is semi-analytic and ϕ is p-free then $\bar{\psi}_{js}$, $\bar{\theta}_{js}$ and $\bar{\phi}_{ir}$ are also p-free, we can use the induction hypothesis to get (for $i \neq 1$):

$$\exists p T_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \quad , \quad \exists p S_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \quad , \quad \bar{\phi}_{1r}, \bar{Y}_1 \Rightarrow \forall p S_{1r}$$

If we let $\{\exists p T_j, \bar{X}_j\}$ and $\{\exists p S_i, \bar{Y}_i\}$ and $\{\bar{Y}_1\}$ and $\{\forall p S_{1r}\}$ be the contexts in the original left semi-analytic rule, we have the following

$$\frac{\langle\langle \exists p T_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \exists p S_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \bar{\phi}_{1r}, \bar{Y}_1 \Rightarrow \forall p S_{1r}\rangle_r}{\exists p T_1, \cdots, \exists p T_m, \exists p S_2, \cdots, \exists p S_n, \bar{X}, \bar{Y}, \phi \Rightarrow \forall p S_1}$$

Using the rule $(L*)$ we have

$$(\mathop{\circledast}_j \exists p T_j) * (\mathop{\circledast}_{i\neq 1} \exists p S_i), \bar{X}, \bar{Y}, \phi \Rightarrow \forall p S_1.$$

Therefore using the rule $(R \to)$, we have

$$\bar{X}, \bar{Y}, \phi \Rightarrow (\mathop{\circledast}_j \exists p T_j) * (\mathop{\circledast}_{i\neq 1} \exists p S_i) \to \forall p S_1.$$

Since the right side of the sequent is a disjunct in the definition of $\forall p U$, using the rule $(R\vee)$ we have $\bar{C}, \phi \Rightarrow \forall p U$.
In the case that $T_1, \cdots, T_m, S_1, \cdots, S_n$ is a trivial partition of U, it means that either $S_1 = U$ or $U^s = \varnothing$ and one of the others is equal to U. The latter case is investigated in the previous case, so it only remains to consider the first one.
If $S_1 = U = \Gamma \Rightarrow \Delta$, then all the others are the empty sequents. Then we must have had the following instance of the rule:

$$\frac{\langle\langle \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \rangle_s\rangle_j \qquad \langle\langle \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma, \phi_{1r}, \bar{Y}_1 \Rightarrow \Delta \rangle_r}{\Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

Therefore, $U \cdot (\phi_{1r}, \bar{Y}_1 \Rightarrow)$ for every r are premises of $U \cdot (\bar{C} \Rightarrow)$, and hence the length of their trees are smaller than the length of the proof tree of $U \cdot (\bar{C} \Rightarrow)$ and since the rule is semi-analytic and ϕ is p-free then $\bar{\phi}_{1r}$ are also p-free, which means that for all of them we can use the induction hypothesis (induction on the length of the proof), and we have $(\phi_{1r}, \bar{Y}_1 \Rightarrow \forall pU)$. Substituting $\{\forall pU\}$, $\{\bar{X}_i\}$ and $\{\bar{Y}_i\}$ as the contexts of the premises in the original left rule and letting all the other contexts in the original left rule to be empty we have

$$\frac{\langle\langle \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \rangle_s\rangle_j \qquad \langle\langle \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \rangle_r\rangle_{i\neq 1} \qquad \langle \phi_{1r}, \bar{Y}_1 \Rightarrow \forall pU \rangle_r}{\bar{X}, \bar{Y}, \phi \Rightarrow \forall pU}$$

which is what we wanted.

- Consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a left semi-analytic rule and $\phi \notin \bar{C}$. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$ is the conclusion of a left semi-analytic rule and U is of the form $\Pi, \Gamma, \phi \Rightarrow \Delta$ and $\bar{C} = \bar{X}, \bar{Y}$ and we want to prove $\bar{X}, \bar{Y} \Rightarrow \forall pU$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r\rangle_i}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta} \quad (\ddagger)$$

where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$, $\bigcup_i \bar{Y}_i = \bar{Y}$ and $\bigcup_i \Delta_i = \Delta$. Since, $\bar{X}_j$'s and $\bar{Y}_i$'s are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take $\bar{X}_j$'s and $\bar{Y}_i$'s in the contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r\rangle_i}{\Pi, \Gamma, \phi \Rightarrow \Delta}$$

If we let $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$, we can claim that this rule is backward applicable to U and T_{js}'s and S_{ir}'s are the premises of the rule. Hence, their orders are less than the order of U and we can use the induction hypothesis for them. Note that we have $U \cdot (\bar{C} \Rightarrow)$ is provable in $\mathbf{H}$ and from (‡) we have that $T_{js} \cdot (\bar{X}_j \Rightarrow)$ and $S_{ir} \cdot (\bar{Y}_i \Rightarrow)$ are also provable in $\mathbf{H}$. Using the induction hypothesis we get

$$\bar{X}_j \Rightarrow \forall p T_{js} \quad , \quad \bar{Y}_i \Rightarrow \forall p S_{ir}$$

Using the rule $(R\wedge)$ we get

$$\bar{X}_j \Rightarrow \bigwedge_s \forall p T_{js} \quad , \quad \bar{Y}_i \Rightarrow \bigwedge_r \forall p S_{ir}$$

and using the rule $(R*)$ we get

$$\bar{X}, \bar{Y} \Rightarrow (\mathop{\circledast}_j \bigwedge_s \forall p T_{js}) * (\mathop{\circledast}_r \bigwedge_r \forall p S_{ir}).$$

Since the right side of the sequent is appeared as the second disjunct in the definition of $\forall p U$, using the rule $(R\vee)$ we have $\bar{C} \Rightarrow \forall p U$.

- Consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a right semi-analytic rule. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Gamma, \bar{C} \Rightarrow \phi)$ is the conclusion of a right semi-analytic rule and U is of the form $\Gamma \Rightarrow \phi$ and we want to prove $\bar{C} \Rightarrow \forall p U$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma, \bar{C} \Rightarrow \phi} \quad (\star)$$

where $\bigcup_i \Gamma_i = \Gamma$ and $\bigcup_i \bar{C}_i = \bar{C}$.
With the similar reasoning as in the previous case, since $\bar{C}_i$'s are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take $\bar{C}_i$'s in the contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma \Rightarrow \phi}$$

If we let $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir})$ we can claim that this rule is backward applicable to U and S_{ir}'s are the premises of the rule. Hence, their orders are less than the order of U and hence we can use the induction hypothesis for them. Using the induction hypothesis we get for every i and r,

$$\bar{C}_i \Rightarrow \forall p S_{ir}.$$

Using the rule $(R\wedge)$ we get $\bar{C}_i \Rightarrow \bigwedge_r \forall p S_{ir}$ and then using the rule $(R*)$ we get $\bar{C}_i \Rightarrow \mathop{\ast}_i \bigwedge_r \forall p S_{ir}$. And since the right side of the sequent is appeared as one of the disjuncts in the definition of $\forall pU$, using the rule $(R\vee)$ we have $\bar{C} \Rightarrow \forall pU$.

- And the final case is when the last rule used in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a semi-analytic modal rule. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box\Delta})$ is the conclusion of a semi-analytic modal rule and U is of the form $(\Box\Gamma \Rightarrow \overline{\Box\Delta})$ and $\bar{C} = \overline{\Box C'}$, where $|\overline{\Box\Delta}| \leqslant 1$ and $U' = (\Gamma \Rightarrow \Delta)$. We want to prove $(\bar{C} \Rightarrow \forall pU)$. We must have had the following instance of the rule

$$\frac{\Gamma, \bar{C'} \Rightarrow \bar{\Delta}}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box\Delta}}$$

Since the order of U' is less than the order of U and C' is p-free, we can use the induction hypothesis and get

$$\bar{C'} \Rightarrow \forall pU'$$

Using the rule K or D (depending on the cardinality of $\overline{\Box\Delta}$) we have $\overline{\Box C'} \Rightarrow \Box\forall pU'$ and since we have $\Box\forall pU'$ as one of the disjuncts in the definition of $\forall pU$, we conclude $\bar{C} \Rightarrow \forall pU$ using the rule $(R\vee)$.

□

Theorem 5.5. *Any terminating single-conclusion sequent calculus* **H** *that extends* $\mathbf{FL_e}$ *and consists of focused axioms and single-conclusion semi-analytic rules, has* **H***-uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 5.3 and Theorem 5.4. □

Corollary 5.6. *If* $\mathbf{FL_e} \subseteq L$ *and* L *has a terminating single-conclusion sequent calculus consisting of focused axioms and single-conclusion semi-analytic rules, then* L *has uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 5.5 and Theorem 5.2. □

In the following application, we will use the Corollary 5.6 to generalize the result of [1] to also cover the modal cases:

Corollary 5.7. *The logics* $\mathbf{FL_e}$*,* $\mathbf{FL_{ew}}$ *and their* $\mathbf{K}$ *and* $\mathbf{KD}$ *versions have uniform interpolation.*

Proof. Since all the rules of the usual calculi of these logics are semi-analytic and their axioms are focused and since in the absence of the contraction rule the calculi are clearly terminating, by Corollary 5.6, we can prove the claim. □

5.1.2 Context-Sharing Semi-analytic Case

In this subsection we will modify the investigations of the last subsection to also cover the context-sharing semi-analytic rules.

Theorem 5.8. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be two single-conclusion sequent calculi with the property that the right and left weakening rules and the context-sharing* $(L \to)$ *rule are admissible in* $\mathbf{H}$ *and* $\mathbf{H}$ *extends* $\mathbf{FL_e}$*. Then if* $\mathbf{H}$ *is a terminating sequent calculus axiomatically extending* $\mathbf{G}$ *with single-conclusion semi-analytic rules and context-sharing semi-analytic rules and* $\mathbf{G}$ *has weak* $\mathbf{H}$*-uniform interpolation property, so does* $\mathbf{H}$*.*

Proof. The proof is similar to the proof of Theorem 5.4. For any sequent U and V where $V^s = \varnothing$ and any atom p, we define two p-free formulas, denoted by $\forall p U$ and $\exists p V$ and we will prove that they meet the conditions in the definition of weak $\mathbf{H}$-uniform interpolation. We define them simultaneously and the definition uses recursion on the rank of sequents which is specified by the terminating condition of the sequent calculus $\mathbf{H}$.

If V is the empty sequent we define $\exists p V$ as 1 and otherwise, we define $\exists p V$ as the following:

$$\bigwedge_{L\mathcal{R}_{cs}} (\mathop{\ast}_{i \neq 1} [(\bigwedge_r (\exists p \tilde{S}_{ir} \to \forall p S_{ir})) \wedge (\bigwedge_s (\exists p \tilde{T}_{is} \to \forall p T_{is}))] * ((\bigwedge_s \exists p \tilde{T}_{1s} \to \forall p T_{1s}) \to \bigvee_r \exists p S_{1r})$$

$$\wedge \bigwedge_{L\mathcal{R}_{cs}} (\mathop{\circledast}_i \bigwedge_r (\exists p\tilde{S}_{ir} \to \forall p S_{ir}) * (\mathop{\circledast}_j \bigwedge_s (\exists p\tilde{T}_{js} \to \forall p T_{js}) \to \bigvee_r \exists p S_{1r})$$

$$\wedge (\bigwedge_{par} \mathop{\circledast}_i \exists p S_i) \wedge (\Box \exists p V') \wedge (\exists^G p V).$$

where for any sequent R, by $\tilde{R}$ we mean $R^a \Rightarrow$. In the first conjunct (the first line), the first big conjunction is over all context-sharing semi-analytic rules that are backward applicable to V in **H**. Since **H** is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{is}\rangle_s\rangle_i$, $\langle\langle S_{ir}\rangle_r\rangle_{i\neq 1}$ and $\langle S_{1r}\rangle$ and the conclusion is V, where $T_{is} = (\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$ which means that S_{ir}'s are those who have context in the right side of the sequents (Δ_i) in the premises of the context-sharing semi-analytic rule. (Note that picking the block $\langle S_{1r}\rangle$ between the S_{ir} blocks is arbitrary and for any choice of $\langle S_{1r}\rangle$, we add one conjuct to the definition.)

In the second conjunct (the second line), the first big conjunction is over all left semi-analytic rules that are backward applicable to V in **H**. Since **H** is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{js}\rangle_s\rangle_j$, $\langle\langle S_{ir}\rangle_r\rangle_{i\neq 1}$ and $\langle S_{1r}\rangle$ and the conclusion is V, where $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$ which means that S_{ir}'s are those who have context in the right side of the sequents (Δ_i) in the premises of the left semi-analytic rule. (Again note that picking the block $\langle S_{1r}\rangle$ between the S_{ir} blocks is arbitrary and for any choice of $\langle S_{1r}\rangle$, we add one conjuct to the definition.)

In the third conjunct (first one in the third line), the conjunction is over all non-trivial partitions of $V = S_1 \cdot \cdots \cdot S_n$ and i ranges over the number of S_i's, in this case $1 \leqslant i \leqslant n$.

The conjunct $\Box\exists p V'$ appears in the definition whenever V is of the form $(\Box\Gamma \Rightarrow)$ and we consider V' to be $(\Gamma \Rightarrow)$. And finally, since **G** has weak **H**-uniform interpolation property, by definition there exist $J(V)$ as weak right p-interpolant of V. We choose one such $J(V)$ and denote it as $\exists^G p V$ and include it in the definition.

If U is the empty sequent define $\forall p U$ as 0. Otherwise, define $\forall p U$ as the following:

$$\bigvee_{L\mathcal{R}_{cs}} (\mathop{\ast}_{i} [\bigwedge_{r} (\exists p\tilde{S}_{ir} \to \forall p S_{ir}) \wedge \bigwedge_{s} (\exists p\tilde{T}_{is} \to \forall p T_{is})])$$

$$\vee \bigvee_{L\mathcal{R}_{cs}} ([\mathop{\ast}_{i} \bigwedge_{r} (\exists p\tilde{S}_{ir} \to \forall p S_{ir})] \ast [\mathop{\ast}_{j} \bigwedge_{s} (\exists p\tilde{T}_{js} \to \forall p T_{js})])$$

$$\vee (\bigvee_{R\mathcal{R}} (\mathop{\ast}_{i} \bigwedge_{r} (\exists p\tilde{S}_{ir} \to \forall p S_{ir})))$$

$$\vee \bigvee_{par} (\mathop{\ast}_{i \neq 1} (\exists p S_i) \to \forall p S_1) \vee (\Box(\exists p\tilde{U'} \to \forall p U')) \vee (\forall^G p U).$$

In the first conjunct (the first line), the first big conjunction is over all context sharing semi-analytic rules that are backward applicable to V in **H**. Since **H** is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{is}\rangle_s\rangle_i$, $\langle\langle S_{ir}\rangle_r\rangle_i$ and the conclusion is V, where $T_{is} = (\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$.

In the second conjunct (the second line), the first big conjunction is over all left semi-analytic rules that are backward applicable to V in **H**. Since **H** is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{js}\rangle_s\rangle_j$, $\langle\langle S_{ir}\rangle_r\rangle_i$ and the conclusion is V, where $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$.

In the third disjunct (the third line), the big disjunction is over all right semi-analytic rules backward applicable to U in **H**. The premise of the rule is $\langle\langle S_{ir}\rangle_r\rangle_i$ and the conclusion is U.

In the fourth disjunct, the big disjunction is over all partitions of $U = S_1 \cdot \cdots \cdot S_n$ such that for each $i \neq 1$ we have $S_i^s = \varnothing$ and $S_1 \neq U$. (Note that in this case, if $S^s = \varnothing$ it may be possible that for one $i \neq 1$ we have $S_i = U$. Then the first disjunct of the definition must be $\exists p U \to \forall p S_1$ where $\forall p S_1 = 0$. But this does not make any problem, since the definition of $\exists p U$ is prior to the definition of $\forall p U$.)

The fifth disjunct is on all semi-analytic modal rules with the result U and the premise U'. And finally, since **G** has weak **H**-uniform interpolation property, by definition there exist $I(U)$ as left weak p-interpolant of U. We choose one such $I(U)$ and denote it as $\forall^G pU$ and include it in the definition.

To prove the theorem we use induction on the order of the sequents to prove both cases $\forall pU$ and $\exists pV$ simultaneously. First note that both $\forall pU$ and $\exists pV$ are p-free by construction and since in all the rules the variables in the premises also occurs in the consequence, we have $V(\forall pU) \subseteq V(U^a) \cup V(U^s)$ and $V(\exists pV) \subseteq V(V^a)$. Secondly, we have to show that:

(i) $V \cdot (\Rightarrow \exists pV)$ is derivable in **H**.

(ii) $U \cdot (\forall pU \Rightarrow)$ is derivable in **H**.

The proof is similar to the proof of the Theorem 5.4. Therefore, we will prove two cases, one for (i) and one for (ii), where there is a notable difference.

- In proving (i), we have to show that $V \cdot (\Rightarrow X)$ is derivable in **H** for any X that is one of the conjuncts in the definition of $\exists pV$. Then, using the rule $(R\wedge)$ it follows that $V \cdot (\Rightarrow \exists pV)$. Since V is of the form $\Gamma \Rightarrow$, we have to show that $\Gamma \Rightarrow X$ is derivable in **H**.
  Consider the case where X is the first conjunct in the definition of $\exists pV$. In this case, we have to prove that for any context-sharing semi-analytic rules that is backward applicable to V in **H**, we have $V \cdot (\Rightarrow Y)$ in **H**, where $X = \bigwedge_{L\mathcal{R}_{cs}} Y$. Therefore, V is the conclusion of a context-sharing semi-analytic rule and is of the form $(\Gamma, \phi \Rightarrow)$ such that the premises are $\langle\langle T_{is}\rangle_s\rangle_i$ and $\langle\langle S_{ir}\rangle_r\rangle_i$, where T_{is} is of the form $(\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and S_{ir} is of the form $(\Gamma_i, \bar{\phi}_{ir} \Rightarrow)$ and we have $\{\Gamma_1, \cdots, \Gamma_n\} = \Gamma$. Therefore, their orders are less than the order of V. Moreover, since $\tilde{T}_{is} = (T^a_{is} \Rightarrow)$ and $\tilde{S}_{ir} = (T^a_{ir} \Rightarrow)$ and they are subsequents of T_{is} and S_{ir}, their orders are less than or equal to the orders of T_{is} and S_{ir}. Hence, we can use the induction hypothesis for all of them.

  Using the induction hypothesis for T_{is}, $\tilde{T}_{is}$, S_{ir} and $\tilde{S}_{ir}$, for $i \neq 1$, we have the following

$$\Gamma_i, \bar{\psi}_{is}, \forall pT_{is} \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma_i, \bar{\psi}_{is} \Rightarrow \exists p\tilde{T}_{is},$$

$$\Gamma_i, \bar{\phi}_{ir}, \forall pS_{ir} \Rightarrow \quad , \quad \Gamma_i, \bar{\phi}_{ir} \Rightarrow \exists p\tilde{S}_{ir}.$$

  And using the induction hypothesis for S_{1r}, T_{1s} and $\tilde{T}_{1s}$ we have

$$\Gamma_1, \bar{\phi}_{1r} \Rightarrow \exists p S_{1r} \quad , \quad \Gamma_1, \bar{\psi}_{1s}, \forall p T_{1s} \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma_1, \bar{\psi}_{1s} \Rightarrow \exists p \tilde{T}_{1s}.$$

Now, using the left context-sharing implication rule, we have

$$\Gamma_i, \bar{\psi}_{is}, \exists p \tilde{T}_{is} \to \forall p T_{is} \Rightarrow \bar{\theta}_{is}$$

$$\Gamma_i, \bar{\phi}_{ir}, \exists p \tilde{S}_{ir} \to \forall p S_{ir} \Rightarrow$$

$$\Gamma_1, \bar{\psi}_{1s}, \exists p \tilde{T}_{1s} \to \forall p T_{1s} \Rightarrow \bar{\theta}_{1s}$$

Now, first using the rules $(L\wedge)$ and $(R\vee)$, we have

$$\Gamma_i, \bar{\psi}_{is}, \bigwedge_s (\exists p \tilde{T}_{is} \to \forall p T_{is}) \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma_i, \bar{\phi}_{ir}, \bigwedge_r (\exists p \tilde{S}_{ir} \to \forall p S_{ir}) \Rightarrow$$

$$\Gamma_1, \bar{\psi}_{1s}, \bigwedge_s (\exists p \tilde{T}_{1s} \to \forall p T_{1s}) \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r \exists p S_{1r}.$$

For simplicity, denote $(\exists p \tilde{T}_{is} \to \forall p T_{is})$ as A_{is} and $(\exists p \tilde{S}_{ir} \to \forall p S_{ir})$ as B_{ir}. If we use the rule $(L\wedge)$ again, and the rule left weakening only for S_{1r}, and not changing the rule for T_{1r}, we have

$$\Gamma_i, \bar{\psi}_{is}, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}) \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma_i, \bar{\phi}_{ir}, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}) \Rightarrow$$

$$\Gamma_1, \bar{\psi}_{1s}, \bigwedge_s A_{1s} \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma_1, \bar{\phi}_{1r}, \bigwedge_s A_{1s} \Rightarrow \bigvee_r \exists p S_{1r}.$$

Now, it is easy to see that the contexts are sharing and we can substitute the above sequents in the original rule. More precisely, in the original context-sharing semi-analytic rule consider $(\Gamma_i, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}))$ as the context of the premises (as Γ_i's in definition of a context-sharing semi-analytic rule 3.2) for $i \neq 1$ and consider $(\Gamma_1, \bigwedge_s A_{1s})$ as the context of the premises for $i = 1$ (as Γ_1's in definition of a context-sharing semi-analytic rule 3.2). Therefore, after substituting the above sequents in the original context-sharing semi-analytic rule, we conclude

$$\Gamma_1, \bigwedge_s A_{1s}, \Gamma_2, \cdots, \Gamma_n, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir})_{i \neq 1}, \phi \Rightarrow \bigvee_r \exists p S_{1r}$$

And finally, using the rule $L*$ and $R\to$ we get

$$\Gamma, \phi \Rightarrow (\circledast_{i\neq 1}(\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}) * (\bigwedge_s A_{1s}) \to \bigvee_r \exists p S_{1r})$$

and this is what we wanted.

- To prove (ii), we have to show that $U \cdot (X \Rightarrow)$ is derivable in **H** for any X that is one of the disjuncts in the definition of $\forall p U$. Then, using the rule $(L\vee)$ it follows that $U \cdot (\forall p U \Rightarrow)$. Since U is of the form $(\Gamma \Rightarrow \Delta)$, we have to show that $(\Gamma, X \Rightarrow \Delta)$ is derivable in **H**.
  In the case that the disjunt is:

  $$\bigvee_{L\mathcal{R}_{cs}} (\circledast_i [\bigwedge_r (\exists p \tilde{S}_{ir} \to \forall p S_{ir}) \wedge \bigwedge_s (\exists p \tilde{T}_{is} \to \forall p T_{is})]),$$

  we have to prove that for any context-sharing semi-analytic rule that is backward applicable to U in **H** we have

  $$U \cdot (\circledast_i [\bigwedge_r (\exists p \tilde{S}_{ir} \to \forall p S_{ir}) \wedge \bigwedge_s (\exists p \tilde{T}_{is} \to \forall p T_{is})] \Rightarrow).$$

  The proof goes exactly as in the previous case (in proof of (i) for context-sharing semi-analytic rules), except that this time the succedents of S_{ir}'s and U are not empty and Δ_i's and Δ appear in their positions everywhere. And, we do not separate the cases T_{1s} and S_{1r} and we proceed with the proof considering the induction hypothesis for every i, in a uniform manner.

Note that these two cases were the cases for the only rule that is not considered in the proof of 5.4. For the proof of (i) for the other conjuncts and (ii) for the other disjuncts, we proceed with the proof of the corresponding cases as in the proof of 5.4, this time substituting $(\exists p \tilde{T}_{js} \to \forall p T_{js})$ for $\forall p T_{js}$ and $(\exists p \tilde{S}_{ir} \to \forall p S_{ir})$ for $\forall p S_{ir}$ wherever it is needed. One can easily see that the proof essentially goes as before, considering this minor change.

Secondly, we have to prove the following, as well.

(iii) For any p-free multisets Γ and Δ, if $T \cdot (\Gamma \Rightarrow \Delta)$ is derivable in **G** then $J(T), \Gamma \Rightarrow \Delta$ is derivable in **H**.

(*iv*) For any p-free multiset Γ, if $S \cdot (\Gamma \Rightarrow)$ is derivable in $\mathbf{G}$ then $J(\tilde{S}), \Gamma \Rightarrow I(S)$ is derivable in $\mathbf{H}$.

Again, since the spirit of the proof is the same as the proof of Theorem 5.4, we will prove two cases for the context-sharing semi-analytic rule, which were not present in the Theorem 5.4. We will prove (*iii*) and (*iv*) simultaneously using induction on the length of the proof and induction on the order of U and V as in the Theorem 5.4.

- To prove (*iii*), consider the case where the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a context-sharing semi-analytic rule and $\phi \notin \bar{C}$. Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{C}, \phi \Rightarrow \Delta)$ is the conclusion of a context-sharing semi-analytic rule and V is of the form $(\Gamma, \phi \Rightarrow)$ and we want to prove $(\exists p V, \bar{C} \Rightarrow \Delta)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle \Gamma_i, \bar{C}_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is} \rangle_s \rangle_i \qquad \langle\langle \Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \rangle_r \rangle_{i \neq 1} \qquad \langle \Gamma_1, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \Delta \rangle_r}{\Gamma, \bar{C}, \phi \Rightarrow \Delta}$$

  where $\bigcup\limits_j \Pi_j = \Pi$, $\bigcup\limits_i \Gamma_i = \Gamma$ and $\bigcup\limits_i \bar{C}_i = \bar{C}$.
  Since, $\bar{C}_i$'s are in the context positions in the original rule, we can consider the same substition of meta-sequents and meta-formulas as above in the original rule, except that we do not take $\bar{C}_i$'s as contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is} \rangle_s \rangle_i \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \rangle_r \rangle_{i \neq 1} \qquad \langle \Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta \rangle_r}{\Gamma, \phi \Rightarrow \Delta}$$

  If we let $T_{is} = (\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow)$ for $i \neq 1$ and $S_{1r} = (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta)$, we can claim that this rule is backward applicable to V and T_{is}'s and S_{ir}'s are the premises of the rule. Hence, their orders are less than the order of V and we can use the induction hypothesis for them. Furthermore, since $\tilde{T}_{is} = (T^a_{is} \Rightarrow)$ and $\tilde{S}_{ir} = (S^a_{ir} \Rightarrow)$, their orders are smaller than or equal to the orders of T_{is} and S_{ir} and we can use the induction hypothesis for them, as well. Using the induction hypothesis (informally speaking, for the first two premises, use the induction hypothesis of $\forall$, and for the last premise use the induction hypothesis of $\exists$) we get

$$(\bar{C}_i, \exists p\tilde{T}_{is} \Rightarrow \forall pT_{is}) \quad , \quad (\bar{C}_i, \exists p\tilde{S}_{ir} \Rightarrow \forall pS_{ir})_{i\neq 1} \quad , \quad (\bar{C}_1, \exists pS_{1r} \Rightarrow \Delta)$$

Now, first using the rules $(R \rightarrow)$ and then using the rule $(R\wedge)$ and $(L\vee)$ we have

$$(\bar{C}_i \Rightarrow \bigwedge_s (\exists p\tilde{T}_{is} \rightarrow \forall pT_{is}))$$

$$(\bar{C}_i \Rightarrow \bigwedge_r (\exists p\tilde{S}_{ir} \rightarrow \forall pS_{ir}))_{i\neq 1}$$

$$(\bar{C}_1, \bigvee_r \exists pS_{1r} \Rightarrow \Delta)$$

Denote $(\bigwedge_s \forall pT_{js})$ as A_j and $(\bigwedge_r \forall pS_{ir})$ as B_i (for $i \neq 1$) and $(\bigvee_r \exists pS_{1r})$ as D. We have for $i \neq 1$

$$\bar{C}_i \Rightarrow A_i \quad , \quad \bar{C}_i \Rightarrow B_i$$

and for $i = 1$ we have

$$\bar{C}_1 \Rightarrow A_1 \quad , \quad \bar{C}_1, D \Rightarrow \Delta.$$

Now, and using the rule $(R\wedge)$ for $i \neq 1$ we get $\bar{C}_i \Rightarrow A_i \wedge B_i$. Together with $\bar{C}_1 \Rightarrow A_1$ and using the rule $(R*)$ we get

$$\bar{C}_1, \bar{C}_2, \cdots, \bar{C}_n \Rightarrow \mathop{\circledast}_i (A_i \wedge B_i) * A_1.$$

Consider the sequent $\bar{C}_1, D \Rightarrow \Delta$ and use the left weakening rule to get

$$\bar{C}_1, \bar{C}_2, \cdots, \bar{C}_n, D \Rightarrow \Delta.$$

Now, use the rule left context-sharing implication to reach

$$\bar{C}, (\mathop{\circledast}_i (A_i \wedge B_i) * A_1) \rightarrow D \Rightarrow \Delta.$$

And, we are done.

- For the proof of (iv), consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a context-sharing semi-analytic rule and $\phi \in \bar{C}$. Therefore,

$$U \cdot (\bar{C} \Rightarrow) = \Gamma, \bar{X}, \phi \Rightarrow \Delta$$

is the conclusion of a context-sharing semi-analytic rule and U is of the form $\Gamma \Rightarrow \Delta$ and $\bar{C} = \bar{X}, \phi$ and we want to prove $\exists p\tilde{U}, \bar{X}, \phi \Rightarrow \forall pU$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Gamma_i, \bar{X}_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is} \rangle_s\rangle_i \qquad \langle\langle \Gamma_i, \bar{X}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r\rangle_i}{\Gamma, \bar{X}, \phi \Rightarrow \Delta}$$

where $\bigcup\limits_i \Gamma_i = \Gamma$, $\bigcup\limits_j \bar{X}_j = \bar{X}$, and $\bigcup\limits_i \Delta_i = \Delta$. Consider $T_{is} = (\Gamma_i \Rightarrow)$, $S_{1r} = (\Gamma_1 \Rightarrow \Delta_1)$, and for $i \neq 1$ let $S_{ir} = (\Gamma_i \Rightarrow)$. Since T_{is}'s do not depend on the suffix s, we have $T_{i1} = \cdots = T_{is}$ and we denote it by T_i. And, since S_{ir}'s do not depend on r for $i \neq 1$, we have $S_{21} = \cdots = S_{ir}$ and we denote it by S_i and with the same line of reasoning we denote S_{1r} by S_1. Therefore, $S_1, \cdots, S_n$ is a partition of U. First, consider the case that $S_1 \neq U$. Then the order of all of them are less than the order of U (or in some cases that one of the others equals to U, the length of the proof is shorter) and since the rule is context sharing semi-analytic and ϕ is p-free then $\bar{\psi}_{is}$ and $\bar{\phi}_{ir}$ are also p-free, we can use the induction hypothesis to get (for $i \neq 1$):

$$\exists pT_i, \bar{\psi}_{is}, \bar{X}_i \Rightarrow \bar{\theta}_{is} \quad , \quad \exists pS_i, \bar{\phi}_{ir}, \bar{X}_i \Rightarrow \quad , \quad \exists p\tilde{S}_1, \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \forall pS_1$$

Note that for every $i \neq 1$ we have $T_i = S_i$ and for $i = 1$ we have $T_1 = \tilde{S}_1$ and we can rewrite the above sequents according to this new information. Hence, if we let $\{\exists pT_i, \bar{X}_i\}$ and $\{\forall pS_1\}$ be the contexts in the original left semi-analytic rule, we have the following

$$\frac{\langle\langle \exists pT_i, \bar{\psi}_{is}, \bar{X}_i \Rightarrow \bar{\theta}_{is} \rangle_s\rangle_i \qquad \langle\langle \exists pT_i, \bar{\phi}_{ir}, \bar{X}_i \Rightarrow \rangle_r\rangle_{i\neq 1} \qquad \langle \exists pT_1, \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \forall pS_1 \rangle_r}{\exists pT_1, \cdots, \exists pT_n, \bar{X}, \phi \Rightarrow \forall pS_1}$$

Using first the rule $(L*)$ and second the rule $R \to$ we get

$$\exists pT_1, \bar{X}, \phi \Rightarrow \mathop{\ast}_{i\neq 1} \exists pT_i \to \forall pS_1$$

Since $T_2, \cdots, T_n, S_1$ is a partition of U, the right hand side of the above sequent is appeared as one of the disjuncts in the definition of $\forall pU$. And since $T_1 = \tilde{U}$, we have

$$\exists p\tilde{U}, \bar{C} \Rightarrow \forall pU$$

and we are done.

We have to investigate the case when $S_1 = U$, as well. However, the line of reasoning is as above and as in the case of $\forall pU$, and $\phi \in \bar{C}$ in the proof of the Theorem 5.4. The important thing is that in the case where $S_1 = U$, with similar reasoning as above, at the end we get $\exists p\tilde{S}_1, \bar{C} \Rightarrow \forall pS_1$ which solves the problem. Note that this case is one of the main reasons that we have changed uniform interpolation to weak uniform interpolation.

And finally, to prove (iii) and (iv) for the other cases, use similar reasoning as in the proof of Theorem 5.4, this time substituting $(\exists p\tilde{T}_{js} \rightarrow \forall pT_{js})$ for $\forall pT_{js}$ and $(\exists p\tilde{S}_{ir} \rightarrow \forall pS_{ir})$ for $\forall pS_{ir}$ wherever it is needed, then the proof easily follows. □

Theorem 5.9. *Any terminating single-conclusion sequent calculus* **H** *that extends* **IPC** *and consists of focused axioms, single-conclusion semi-analytic and context-sharing semi-analytic rules, has weak* **H***-uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 5.3 and the Theorem 5.8. □

Corollary 5.10. *If* **IPC** $\subseteq L$ *and* L *has a terminating single-conclusion sequent calculus consisting of focused axioms, single-conclusion semi-analytic rules and context-sharing semi-analytic rules, then* L *has uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 5.9 and the Theorem 5.2. □

Definition 5.11. We will define the following sequent calculus for intuitionistic logic, **G4i**, which was first introduced by Dyckhoff in [4].

$$\phi \Rightarrow \phi \quad (Id) \quad , \quad \Gamma, \bot \Rightarrow \Delta \quad (L\bot) \quad , \quad \Gamma \Rightarrow \Delta, \top \quad (R\top)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\,(Lw) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \phi}\,(Rw)$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta} \, (L\wedge) \qquad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta} \, (R\wedge)$$

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta} \, (L\vee) \qquad \frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi} \, (R\vee) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi} \, (R\vee)$$

$$\frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi} \, (R\to)$$

$$\frac{\Gamma, p, \psi \Rightarrow \Delta}{\Gamma, p, p \to \psi \Rightarrow \Delta} \, (L_1\to) \qquad \frac{\Gamma, \phi \to (\psi \to \gamma) \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \to \gamma \Rightarrow \Delta} \, (L_2\to)$$

$$\frac{\Gamma, \phi \to \gamma, \psi \to \gamma \Rightarrow \Delta}{\Gamma, \phi \vee \psi \to \gamma \Rightarrow \Delta} \, (L_3\to) \qquad \frac{\Gamma, \psi \to \gamma \Rightarrow \phi \to \psi \qquad \Gamma, \gamma \Rightarrow \Delta}{\Gamma, (\phi \to \psi) \to \gamma \Rightarrow \Delta} \, (L_4\to)$$

where p is an atom. Structural rules and the cut rule are admissible in the system and in each rule Δ has at most one element. Note that this system is slightly different than the usual **G4i** system. The usual definition does not include the explicit weakening rules and the axioms for $\top$ and $\bot$. It also has the axiom $\Gamma, p \Rightarrow p$ only for atomic formula p, instead of the axiom (Id) as we assumed. The system we have introduced is clearly equivalent to the usual one and it is also terminating with the same Dyckhoff order [4] that we will see in a moment. The advantage of the new system, though, is that it is more in line with our later general approach to sequent-style rules.

Define the rank of a propositional formula as follows:

$$\begin{aligned} r(p) &= r(\bot) = r(\top) = 1 \\ r(\phi \circ \psi) &= r(\phi) + r(\psi) + 1 \quad \circ \in \{\vee, \to\} \\ r(\phi \wedge \psi) &= r(\phi) + r(\psi) + 2 \end{aligned}$$

Then a sequent S is called lower than the sequent T if S is the result of replacing the elements of T with any number of elements with lower ranks. With this order, it is not hard to see that the system **G4i** is terminating. Note that with this order, for any formula ψ and any atom p, the sequent $\Gamma, \psi \Rightarrow \Delta$ is lower than the sequent $\Gamma, p \to \psi \Rightarrow \Delta$. We will use this fact in Corollary 5.12.

Corollary 5.12. *[11] The logic* **IPC** *has uniform interpolation.*

Proof. Use **G4i**, the Dyckhoff terminating calculus for **IPC**, introduced in the Preliminaries section. Using the Theorem 5.2, it is enough to show that this system has weak **G4i**-uniform interpolation. For this matter, note that all the rules in this calculus, except the rules $(L_4 \to)$ and $(L_1 \to)$ are semi-analytic, while $(L_4 \to)$ is context-sharing semi-analytic and all the axioms are focused. Therefore, the system has only one rule beyond our context-sharing semi-analytic machinery, namely $(L_1 \to)$. However, note that the proof for the Theorem 5.8 is pretty modular which addresses any rule separately by adding its corresponding disjunct or conjunct in the recursive definition of $\forall pS$ and $\exists pS$, respectively. Therefore, to prove the claim it is enough to add other disjunct and conjunct terms to also address the rule $(L_1 \to)$. This is what we will implement in the following:

For $\forall pS$ add the following terms as disjuncts to the definition of $\forall pS$ as defined in the proof of the Theorem 5.8:

- $\forall^1_{at} pS$ For any atom $q \neq p$ if $q \in S^a$ add $q \to (\exists p\tilde{S}' \to \forall pS')$ where S' is S after eliminating one occurrence of q in S^a.
- $\forall^2_{at} pS$ For any atom $q \neq p$ if $q \to \psi \in S^a$ for some formula ψ add $(\exists p\tilde{S}' \to \forall pS') \wedge q$ where S' is S after replacing one occurrence of $q \to \psi$ by ψ in S^a.

And for $\exists pS$ add the following terms as conjuncts:

- $\exists^1_{at} pS$ For any atom $q \neq p$ if $q \in S^a$ add $q \wedge \exists pS'$ where S' is S after eliminating one occurrence of q in S^a.
- $\exists^2_{at} pS$ For any atom $q \neq p$ if $q \to \psi \in S^a$ for some formula ψ add $q \to \exists pS'$ where S' is S after replacing one occurrence of $q \to \psi$ by ψ in S^a.

The first thing to check is that based on the well-founded order on the sequents used for the system **G4i**, the sequent S' in all cases is below the sequent S and hence the recursive step is well-defined. This is clear because in two cases S' is a proper subsequent of S and in two other cases, we are replacing a formula of the form $q \to \psi$ by ψ which has lower rank according to the rank function we introduced in the Preliminaries. Secondly, note that the number of disjuncts or conjuncts that we are adding are clearly finite and hence $\forall pS$ and $\exists pS$ are well-defined as formulas. Finally, note that we are only using $q \neq p$ in the terms and hence $\forall pS$ and $\exists pS$ remain p-free.

Moreover, since in all cases $V(S') \subseteq V(S)$, by induction on the Dyckhoff's order we have $V(\forall pS) \subseteq V(S^a) \cup V(S^s)$ and $V(\exists pS) \subseteq V(S^a)$.

Now we have to check that adding these terms respects the properties that we have discussed in the proof of the Theorem 5.8. First, let us check that adding the disjuncts $\forall^1_{at} pS$ and $\forall^2_{at} pS$ to $\forall pS$ respects the property (ii) namely $\mathbf{G4i} \vdash S \cdot (\forall pS \Rightarrow)$. We have two cases to check:

For $\forall^1_{at} pS$, let us assume that $S = (\Gamma, q \Rightarrow \Delta)$. Then it is enough to prove that $\Gamma, q, q \to (\exists p\tilde{S}' \to \forall pS') \Rightarrow \Delta$ where $S' = (\Gamma \Rightarrow \Delta)$. Using the rule $(L_1 \to)$, it is enough to prove the sequent $\Gamma, q, (\exists p\tilde{S}' \to \forall pS') \Rightarrow \Delta$. But note that by the IH, we have $\Gamma \Rightarrow \exists p\tilde{S}'$ and $\Gamma, \forall pS' \Rightarrow \Delta$. Therefore, by applying $(L \to)$ and weakening by q (both admissible in $\mathbf{G4i}$) we have $\Gamma, q, (\exists p\tilde{S}' \to \forall pS') \Rightarrow \Delta$.

For $\forall^2_{at} pS$, let us assume that $S = (\Gamma, q \to \psi \Rightarrow \Delta)$. Then $S' = (\Gamma, \psi \Rightarrow \Delta)$ and we want to prove that $\Gamma, q \to \psi, (\exists p\tilde{S}' \to \forall pS') \wedge q \Rightarrow \Delta$. Again using the rule $(L_1 \to)$ itself, it is enough to prove $\Gamma, q, \psi, (\exists p\tilde{S}' \to \forall pS') \Rightarrow \Delta$. By IH we have $\Gamma, \psi \Rightarrow \exists p\tilde{S}'$ and $\Gamma, \psi, \forall pS' \Rightarrow \Delta$. By $(L \to)$ and weakening by q (both admissible in $\mathbf{G4i}$), we can prove $\Gamma, q, \psi, (\exists p\tilde{S}' \to \forall pS') \Rightarrow \Delta$.

Now we will show that adding the conjuncts $\exists^1_{at} pS$ and $\exists^2_{at} pS$ to $\exists pS$ respects the property (i) namely $\mathbf{G4i} \vdash S \cdot (\Rightarrow \exists pS)$ for any S such that $S^s = \varnothing$.

For $\exists^1_{at} pS$, let us assume that $S = (\Gamma, q \Rightarrow)$. Then it is enough to prove that $\Gamma, q \Rightarrow q \wedge \exists pS'$ where $S' = (\Gamma \Rightarrow)$. By the IH, we have $\Gamma \Rightarrow \exists pS'$ and hence we have what we wanted by $(\wedge R)$ and weakening by q.

For $\exists^2_{at} pS$ let us assume that $S = (\Gamma, q \to \psi \Rightarrow)$. Then $S' = (\Gamma, \psi \Rightarrow)$ and we want to prove that $\Gamma, q \to \psi \Rightarrow q \to \exists pS'$. Using the rule $(\to R)$, it is enough to prove $\Gamma, q, q \to \psi \Rightarrow \exists pS'$. By $(L_1 \to)$ itself, it is enough to prove $\Gamma, q, \psi \Rightarrow \exists pS'$. But by IH we have $\Gamma, \psi \Rightarrow \exists pS'$ which implies what we wanted.

Now we are ready to check the other conditions, meaning:

(iii) For any p-free multisets $\bar{C}$ and $\bar{D}$, if $S \cdot (\bar{C} \Rightarrow \bar{D})$ is derivable in $\mathbf{G4i}$ then $\exists pS, \bar{C} \Rightarrow \bar{D}$ is derivable in $\mathbf{G4i}$ for any S that $S^s = \varnothing$.

(*iv*) For any p-free multiset $\bar{C}$, if $S \cdot (\bar{C} \Rightarrow)$ is derivable in **G4i** then $\exists p \tilde{S}, \bar{C} \Rightarrow \forall p S$ is derivable in **G4i**.

First let us prove (*iv*). It is enough to address the case that the last rule in the proof of $S \cdot (\bar{C} \Rightarrow)$ is the rule $(L_1 \rightarrow)$. There are four cases to consider:

- Both q and $q \rightarrow \psi$ are in $\bar{C}$. This case is similar to the left semi-analytic case in the proof of the Theorem 5.8 where the main formula is in $\bar{C}$.

- Both q and $q \rightarrow \psi$ are not in $\bar{C}$. This case is similar to the left semi-analytic case in the proof of the Theorem 5.8 where the main formula is not in $\bar{C}$.

- $q \rightarrow \psi \in \bar{C}$ and $q \notin \bar{C}$. Since $q \rightarrow \psi$ is in $\bar{C}$, it is p-free and hence $q \neq p$ and ψ is p-free. We have

$$\frac{\Gamma, q, \psi \Rightarrow \Delta}{\Gamma, q, q \rightarrow \psi \Rightarrow \Delta}$$

  Define $\Gamma' = \Gamma - \bar{C}$ and $\bar{C}' = \bar{C} - \{q \rightarrow \psi\}$. Therefore, $S = (\Gamma', q \Rightarrow \Delta)$. Define $S' = (\Gamma' \Rightarrow \Delta)$. Since both q and ψ are p-free and S' is a proper subsequent of S and hence lower than S in the Dyckhoff's order, by IH, $\exists p \tilde{S}', \bar{C}', q, \psi \Rightarrow \forall p S'$. By $(L_1 \rightarrow)$ we have $\exists p \tilde{S}', \bar{C}', q, q \rightarrow \psi \Rightarrow \forall p S'$ Hence, $\bar{C}', q \rightarrow \psi \Rightarrow q \rightarrow (\exists p \tilde{S}' \rightarrow \forall p S')$. Since the right hand-side is a disjunct in $\forall p S$, we have $q \rightarrow \psi, \bar{C}' \Rightarrow \forall p S$ and by weakening $\exists p \tilde{S}, q \rightarrow \psi, \bar{C}' \Rightarrow \forall p S$.

- $q \rightarrow \psi \notin \bar{C}$ and $q \in \bar{C}$. Since $q \in \bar{C}$, it is not p itself. Again, we have

$$\frac{\Gamma, q, \psi \Rightarrow \Delta}{\Gamma, q, q \rightarrow \psi \Rightarrow \Delta}$$

  Define $\Gamma' = \Gamma - \bar{C}$ and $\bar{C}' = \bar{C} - \{q\}$. Therefore, $S = (\Gamma', q \rightarrow \psi \Rightarrow \Delta)$. Define $S' = (\Gamma', \psi \Rightarrow \Delta)$. Since q is p-free and S' is lower than S in the Dyckhoff's order, by IH, $\exists p \tilde{S}', \bar{C}', q \Rightarrow \forall p S'$. Hence, $\bar{C}', q \Rightarrow (\exists p \tilde{S}' \rightarrow \forall p S') \wedge q$. Since the right hand-side is a disjunct in $\forall p S$, we have $\bar{C}', q \Rightarrow \forall p S$ and by weakening $\exists p \tilde{S}, q, \bar{C}' \Rightarrow \forall p S$. .

For (*iii*), again there are four cases:

- Both q and $q \to \psi$ are in $\bar{C}$. This case is similar to the left semi-analytic case in the proof of the Theorem 5.8 where the main formula is in $\bar{C}$.

- Both q and $q \to \psi$ are not in $\bar{C}$. This case is similar to the left semi-analytic case in the proof of the Theorem 5.8 where the main formula is not in $\bar{C}$.

- $q \to \psi \in \bar{C}$ and $q \notin \bar{C}$. Since $q \to \psi$ is in $\bar{C}$, it is p-free and hence $q \neq p$ and ψ is p-free. We have

$$\frac{\Gamma, q, \psi \Rightarrow \bar{D}}{\Gamma, q, q \to \psi \Rightarrow \bar{D}}$$

  Define $\Gamma' = \Gamma - \bar{C}$ and $\bar{C}' = \bar{C} - \{q \to \psi\}$. Therefore, $S = (\Gamma', q \Rightarrow)$. Define $S' = (\Gamma' \Rightarrow)$. Since both q and ψ are p-free and S' is a proper subsequent of S and hence lower than S in the Dyckhoff's order, by IH, $\exists p S', \bar{C}', q, \psi \Rightarrow \bar{D}$. By $(L_1 \to)$ we have $\exists p S', \bar{C}', q, q \to \psi \Rightarrow \bar{D}$ Hence, $(\exists p S' \wedge q), \bar{C}', q \to \psi \Rightarrow \bar{D}$. Since $(\exists p S' \wedge q)$ is a conjuct in $\exists p S$, we have $\exists p S, q \to \psi, \bar{C}' \Rightarrow \bar{D}$.

- $q \to \psi \notin \bar{C}$ and $q \in \bar{C}$. Since $q \in \bar{C}$, it is not p itself. again, we have

$$\frac{\Gamma, q, \psi \Rightarrow \bar{D}}{\Gamma, q, q \to \psi \Rightarrow \bar{D}}$$

  Define $\Gamma' = \Gamma - \bar{C}$ and $\bar{C}' = \bar{C} - \{q\}$. Therefore, $S = (\Gamma', q \to \psi \Rightarrow)$. Define $S' = (\Gamma', \psi \Rightarrow)$. Since q is p-free and S' is lower than S in the Dyckhoff's order, by IH, $\exists p S', \bar{C}', q \Rightarrow \bar{D}$. Hence by $(L_1 \to)$, we have $\bar{C}', q \to \exists p S', q \Rightarrow \bar{D}$. Since $q \to \exists p S'$ is a conjunct in $\exists p S$, we have $\exists p S, \bar{C}', q \Rightarrow \bar{D}$.

□

5.2 The Multi-conclusion Case

Finally we will move to the multi-conclusion case to handle the more general form of semi-analytic rules.

Theorem 5.13. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be two multi-conclusion sequent calculi and* $\mathbf{H}$ *extends* $\mathbf{CFL_e}$*. Then if* $\mathbf{H}$ *is a terminating sequent calculus axiomatically extending* $\mathbf{G}$ *with multi-conclusion semi-analytic rules and* $\mathbf{G}$ *has strong* $\mathbf{H}$*-uniform interpolation property, so does* $\mathbf{H}$*.*

Proof. For a given sequent $S = (\Gamma \Rightarrow \Delta)$ and an atom p, we define a p-free formula, denoted by $\forall pS$ and we will prove that it meets the conditions for the strong left and right p-interpolants of S, respectively.

If S is the empty sequent define $\forall pS$ as 0. Otherwise, define $\forall pS$ as

$$\bigvee_{\mathcal{R}} (\mathop{\ast}_{i} \bigwedge_{r} \forall pS_{ir}) \vee \bigvee_{par} (\mathop{+}_{i} \forall pS_i) \vee (\Box \forall pS') \vee (\neg \Box \neg \forall pS'') \vee (\forall^G pS)$$

where the first disjunction is over all multi-conclusion semi-analytic rules backward applicable to S in $\mathbf{H}$, which means the result is S and the premises are S_{ir}. Since $\mathbf{H}$ is terminating, there are finitely many of such rules. The second disjunction is over all non-trivial partitions of S. The third disjunction is over all semi-analytic modal rules with the result S and the premise S'. Moreover, If S is of the form $\Box\Gamma \Rightarrow$, then we consider S'' to be $\Gamma \Rightarrow$ and $\neg \Box \neg \forall pS''$ must be appeared in the definition of $\forall pS$. And finally $\forall^G pS$ is the strong left p-interpolant of a sequent S in $\mathbf{G}$ relative to $\mathbf{H}$.

We define the strong right p-interpolant of S as $\neg \forall pS$ and we denote it by $\exists pS$. Note that if we prove $\forall pS$ is the strong left p-interpolant, it is easy to show that $\exists pS$ meets the conditions for the strong right p-interpolant. The reason is the following: First we have to show that $\Gamma \Rightarrow \Delta, \exists pS$ is provable in $\mathbf{H}$. But we have $\Gamma, \forall pS \Rightarrow \Delta$ is provable in $\mathbf{H}$ and using the rule $(0w)$, we have $\Gamma, \forall pS \Rightarrow \Delta, 0$ which means $\Gamma \Rightarrow \Delta, \neg \forall pS$ is provable in $\mathbf{H}$.
Secondly, we have to show that if for p-free multisets Σ and Λ, if $\Gamma, \Sigma \Rightarrow \Lambda, \Delta$ is derivable in $\mathbf{G}$, then $\exists pS, \Sigma \Rightarrow \Lambda$ is derivable in $\mathbf{H}$. However, we have $\Sigma \Rightarrow \Lambda, \forall pS$ is derivable in $\mathbf{H}$ and using the axiom $0 \Rightarrow$ we can use the rule $(L \rightarrow)$ to get $\Sigma, \neg \forall pS \Rightarrow \Lambda$ in $\mathbf{H}$.

Now let us prove that $\forall pS$ meets all the conditions of a strong left p-interpolant. The proof is similar to the proofs of the Theorems 5.4 and 5.8. To prove the theorem we use induction on the order of the sequents. First note that $\forall pS$ is p-free by construction and since in all the rules the variables in the premises also occurs in the consequence, we have $V(\forall pS) \subseteq V(S^a) \cup V(S^s)$. Secondly, we have to show that:

(*i*) $S \cdot (\forall p S \Rightarrow)$ is provable in $\mathbf{H}$.

We have to show that $\Gamma, X \Rightarrow \Delta$ is derivable in $\mathbf{H}$ for every disjunct X in the definition of $\forall p S$.

- In the case that the disjunct is $\bigvee_{\mathcal{R}} (\circledast_i \bigwedge_r \forall p S_{ir})$, we have to show that for any multi-conclusion semi-analytic rule $\mathcal{R}$ with the premises S_{ir} we have
$$S \cdot (\circledast_i \bigwedge_r \forall p S_{ir} \Rightarrow)$$
where S is of the form $(\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n)$ and S_{ir} is of the form $(\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i)$. Note that since $S'_{ir}s$ are the premises of the rule, the order of all of them are less than the order of S and we can use the induction hypothesis for them. We have for every i and r
$$\Gamma_i, \bar{\phi}_{ir}, \forall p S_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i$$
Using the rule $(L\wedge)$ we have for every i
$$\Gamma_i, \bar{\phi}_{ir}, \bigwedge_r \forall p S_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i$$
Using $\Gamma_i, \bigwedge_r \forall p S_{ir}$ as the left context in the original rule (we can do this, since $\bigwedge_r \forall p S_{ir}$ does not depend on r and it only ranges over i), we have
$$\Gamma_1, \cdots, \Gamma_n, \langle \bigwedge_r \forall p S_{ir} \rangle_i, \phi \Rightarrow \Delta_1, \cdots, \Delta_n$$
and then using the rule $(L*)$, we have
$$\Gamma_1, \cdots, \Gamma_n, (\circledast_i \bigwedge_r \forall p S_{ir}), \phi \Rightarrow \Delta_1, \cdots, \Delta_n.$$

- In the case that the disjunct is $\bigvee_{par} \bigplus_i \forall p S_i$, we have to show that for any non-trivial partition $S_1, \cdots, S_n$ of S we have$S \cdot (\bigplus_i \forall p S_i \Rightarrow)$ is derivable in $\mathbf{H}$. Since the order of each S_i is less than the order of S, we can use the induction hypothesis for them and get $(\Gamma_i, \forall p S_i \Rightarrow \Delta_i)$. Using the rule $(L+)$ we get $\Gamma_1, \cdots, \Gamma_n, (\bigplus_i \forall p S_i) \Rightarrow \Delta_1, \cdots, \Delta_n$.

- ∘ The proof of case that the disjunct is $\Box\forall p S'$ is exactly the same as the similar case in the proof of the Theorem 5.4.

- ∘ In the case that the disjunct is $\neg\Box\neg\forall p S''$, the sequent S must have been of the form $(\Box\Gamma \Rightarrow)$ and S'' is of the form $(\Gamma \Rightarrow)$. Since the order of S'' is less than the order of S, we can use the induction hypothesis and get $(\Gamma, \forall p S'' \Rightarrow)$ is derivable in $\mathbf{H}$. Using the rule $(0w)$ and then the rule $(R \rightarrow)$ we have $(\Gamma \Rightarrow \neg\forall p S'')$. Using the rule (K) we have $(\Box\Gamma \Rightarrow \Box\neg\forall p S'')$ and together with the axiom $(0 \Rightarrow)$ we can use the rule $(L \rightarrow)$ and we have $(\Box\Gamma, \neg\Box\neg\forall p S'' \Rightarrow)$ is derivable in $\mathbf{H}$.

- ∘ The case for $\forall^G p S$, holds trivially by definition.

Second, we have to show that

(*ii*) For any p-free multisets $\bar{C}$ and $\bar{D}$, if $S \cdot (\bar{C} \Rightarrow \bar{D})$ is derivable in $\mathbf{G}$ then $\bar{C} \Rightarrow \forall p S, \bar{D}$ is derivable in $\mathbf{H}$.

We will prove it using induction on the length of the proof and induction on the order of S. More precisely, first by induction on the order of S and then inside it, by induction on n, we will show:

- For any p-free multisets $\bar{C}$ and $\bar{D}$, if $S \cdot (\bar{C} \Rightarrow \bar{D})$ has a proof in $\mathbf{G}$ with length less than or equal to n, then $\bar{C} \Rightarrow \forall p S, \bar{D}$ is derivable in $\mathbf{H}$.

First note that for the empty sequent, we have to show that if $\bar{C} \Rightarrow \bar{D}$ is valid in $\mathbf{G}$, then $\bar{C} \Rightarrow 0, \bar{D}$ is valid in $\mathbf{H}$, which is trivial by the rule $(0w)$.

For the base of the other induction, note that if $n = 0$, it means that $\Gamma, \bar{C} \Rightarrow \bar{D}, \Delta$ is valid in $\mathbf{G}$. Therefore, by Definition 5.1, $\bar{C} \Rightarrow \forall^G p S, \bar{D}$ and hence $\bar{C} \Rightarrow \forall p S, \bar{D}$ is valid in $\mathbf{H}$.

For $n \neq 0$ we have to consider the following cases:

- ∘ Consider the case that the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is a left multi-conclusion semi-analytic rule and $\phi \in \bar{C}$ (which means that the main formula of the rule, ϕ, is one of C_i's). Therefore, $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{X}, \phi \Rightarrow \bar{D}, \Delta)$ is the conclusion of the rule and S is of the form $(\Gamma \Rightarrow \Delta)$ and $\bar{C} = (\bar{X}, \phi)$ and we want to prove $(\bar{X}, \phi \Rightarrow \forall p S, \bar{D})$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Gamma_i, \bar{X}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bar{D}_i, \Delta_i \rangle_r \rangle_i}{\Gamma, \bar{X}, \phi \Rightarrow \bar{D}, \Delta}$$

where $\bigcup\limits_i \Gamma_i = \Gamma$, $\bigcup\limits_i \bar{X}_i = \bar{X}$, $\bigcup\limits_i \bar{D}_i = \bar{D}$ and $\bigcup\limits_i \Delta_i = \Delta$. Consider $S_{ir} = (\Gamma_i \Rightarrow \Delta_i)$. Since S_{ir}'s do not depend on the suffix r, all of them are equal and we denote it by S_i. Therefore, $S_1, \cdots, S_n$ is a partition of S. First, consider that it is a non-trivial partition of S. Then the order of all of them are less than the order of S and since the rule is semi-analytic and ϕ is p-free then $\bar{\phi}_{ir}$ and $\bar{\psi}_{ir}$ are also p-free, we can use the induction hypothesis to get for every i and r:

$$\bar{X}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bar{D}_i, \forall p S_i$$

If we let $\bar{X}_i$ and $\bar{D}_i, \forall p S_i$ be the contexts in the left side and right side in the original rule, respectively, we have the following

$$\bar{X}, \phi \Rightarrow \bar{D}, \forall p S_1, \cdots, \forall p S_n$$

Using the rule $(R+)$ we have

$$\bar{X}, \phi \Rightarrow \bar{D}, \bigplus_i \forall p S_i$$

Since the right side of the sequent is a disjunct in the definition of $\forall p U$, using the rule $(R\vee)$ we have $\bar{C}, \phi \Rightarrow \forall p S, \bar{D}$.

In the case that $S_1, \cdots, S_n$ is a trivial partition of S, it means that one of them equals S. W.l.o.g. suppose $S_1 = S$ and all of the others are the empty sequents. Then we must have had the following instance of the rule:

$$\frac{\langle\langle \bar{\phi}_{ir}, \bar{X}_i \Rightarrow \bar{\psi}_{ir}, \bar{D}_i \rangle_r \rangle_{i \neq 1} \qquad \langle \Gamma, \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \bar{\psi}_{1r}, \bar{D}_1, \Delta \rangle_r}{\Gamma, \phi, \bar{X} \Rightarrow \bar{D}, \Delta}$$

Therefore, $S \cdot (\phi_{1r}, \bar{X}_1 \Rightarrow \bar{\psi}_{1r}, \bar{D}_1)$ for every r are premises of $S \cdot (\bar{C} \Rightarrow \bar{D})$, and hence the length of their trees are smaller than the length of the proof tree of $S \cdot (\bar{C} \Rightarrow \bar{D})$ and since the rule is semi-analytic and ϕ is p-free then $\bar{\phi}_{1r}$ and $\bar{\psi}_{1r}$ are also p-free, which means that for all of them we can use the induction hypothesis (induction on the length of the proof), and we have $(\phi_{1r}, \bar{X}_1 \Rightarrow \forall p S, \bar{\psi}_{1r}, \bar{D}_1)$. Substituting $\{\bar{X}_j\}$ and $\{\forall p S, \bar{D}_1\}$ as the contexts of the premises in the original rule we have

$$\frac{\langle\langle \bar{\phi}_{ir}, \bar{X}_i \Rightarrow \bar{\psi}_{ir}, \bar{D}_i \rangle_r \rangle_{i \neq 1} \qquad \langle \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \forall p S, \bar{\psi}_{1r}, \bar{D}_1 \rangle_r}{\bar{X}, \phi \Rightarrow \forall p S, \bar{D}}$$

which is what we wanted.

- Consider the case where the last rule in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is a left multi-conclusion semi-analytic rule and $\phi \notin \bar{C}$. Therefore, $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{C}, \phi \Rightarrow \bar{D}, \Delta)$ is the conclusion of the rule and S is of the form $\Gamma, \phi \Rightarrow \Delta$ and we want to prove $\bar{C} \Rightarrow \forall p S, \bar{D}$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bar{D}_i, \Delta_i \rangle_r \rangle_i}{\Gamma, \bar{C}, \phi \Rightarrow \bar{D}, \Delta}$$

where $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_i \bar{C}_i = \bar{C}$, $\bigcup_i \bar{D}_i = \bar{D}$ and $\bigcup_i \Delta_i = \Delta$.
Since, $\bar{C}_i$'s and $\bar{D}_i$'s are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take $\bar{C}_i$'s and $\bar{D}_i$'s in the contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma, \phi \Rightarrow \Delta}$$

If we let $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i)$, we can claim that this rule is backward applicable to S and S_{ir}'s are the premises of the rule. Hence, their orders are less than the order of S and we can use the induction hypothesis for them. Using the induction hypothesis we get for every i and r

$$\bar{C}_i \Rightarrow \forall p S_{ir}, \bar{D}_i$$

Using the rule $(R\wedge)$ we get for every i

$$\bar{C}_i \Rightarrow \bigwedge_r \forall p S_{ir}, \bar{D}_i$$

and using the rule $(R*)$ we get

$$\bar{C} \Rightarrow \mathop{\Large *}_i \bigwedge_r \forall p S_{ir}, \bar{D}.$$

Since the right side of the sequent is appeared as one of the disjuncts in the definition of $\forall p S$, using the rule $(R\vee)$ we have $\bar{C} \Rightarrow \forall p S, \bar{D}$.

- Consider the case when the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is a semi-analytic modal rule. Therefore, $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box D'})$ is the conclusion of a semi-analytic modal rule. Hence, there are two cases to consider.
  The first one is the case where S is of the form $(\Box\Gamma \Rightarrow)$ and $\bar{C} = \overline{\Box C'}$ and $\bar{D} = \overline{\Box D'}$, where $|\overline{\Box D'}| \leqslant 1$ and $S'' = (\Gamma \Rightarrow)$. We want to prove $(\bar{C} \Rightarrow \forall pS, \bar{D})$. We must have had the following instance of the rule

$$\frac{\Gamma, \bar{C'} \Rightarrow \bar{D'}}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box D'}}$$

  Since the order of S'' is less than the order of S and C' and D' are p-free, we can use the induction hypothesis and get

$$\bar{C'} \Rightarrow \forall pS'', \bar{D'}$$

  Using the axiom $(0 \Rightarrow)$ and the rule $(L \rightarrow)$ we have

$$\bar{C'}, \neg\forall pS'' \Rightarrow \bar{D'}$$

  Now, using the rule K or D (depending on the cardinality of $\bar{D'}$) we have

$$\overline{\Box C'}, \Box\neg\forall pS'' \Rightarrow \overline{\Box D'}$$

  and using the rule $(0w)$ and $(R \rightarrow)$ we get

$$\overline{\Box C'} \Rightarrow \neg\Box\neg\forall pS'', \overline{\Box D'}$$

  since we have $\neg\Box\neg\forall pS''$ as one of the disjuncts in the definition of $\forall pS$, we conclude $\bar{C} \Rightarrow \forall pS, \bar{D}$ using the rule $(R\vee)$.

  The second case is when S is of the form $\Box\Gamma \Rightarrow \Box D'$, where D' is a p-free formula and S' is of the form $\Gamma \Rightarrow D$. We want to prove $\bar{C} \Rightarrow \forall pS$. Then we must have had the following instance of the rule

$$\frac{\Gamma, \bar{C'} \Rightarrow \bar{D'}}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box D'}}$$

Since $\bar{C}'$ is in the context position of the original rule, we can consider the same substitution of meta-sequents as above in the original rule, except that we do not take $\bar{C}'$ in the context. More precisely, we reach the following instance of the original rule:

$$\frac{\Gamma \Rightarrow \bar{D}'}{\Box\Gamma \Rightarrow \overline{\Box D'}}$$

Therefore, this rule is backward applicable to S and the order of the premise, S', is less than the order of S and we can use the induction hypothesis for that to reach $C' \Rightarrow \forall p S'$. Then we can use the rule K and we get $\overline{\Box C'} \Rightarrow \Box \forall p S'$, which is a disjunct in the definition of $\forall p S$ and we have $\bar{C} \Rightarrow \forall p S$.

- The case for the right multi-conclusion semi-analytic rules is similar to the cases for the left ones disccused in this proof, and the proof of other two cases are similar to the proof of the same cases in the Theorem 5.4.

□

Theorem 5.14. *Any terminating multi-conclusion sequent calculus* **H** *that extends* $\mathbf{CFL_e}$ *and consists of focused axioms and multi-conclusion semi-analytic rules, has strong* **H***-uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 5.3 and Theorem 5.13. □

Corollary 5.15. *If* $\mathbf{CFL_e} \subseteq L$ *and* L *has a terminating multi-conclusion sequent calculus consisting of focused axioms and multi-conclusion semi-analytic rules, then* L *has uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 5.14 and Theorem 5.2. □

Using the Theorem 5.15, we can extend the results of [1] and [2] to:

Corollary 5.16. *The logics* $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$ *and* **CPC** *and their* **K** *and* **KD** *modal versions have uniform interpolation property.*

Proof. For $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, since all the rules of the usual calculus of these logics are semi-analytic and their axioms are focused and since in the absence of the contraction rule the calculus is clearly terminating, by Theorem 5.15, we can prove the claim. For **CPC** use the contraction-free calculus for which the proof goes as the other cases. □

In the negative side we use the negative results in [2], [5] and [6] to ensure that the following logics do not have uniform interpolation. Then we will use the Theorems 5.6, 5.10 and 5.15 to the non-existence of terminating calculus consisting only of semi-analytic and context-sharing semi-analytic rules together with focused axioms.

Corollary 5.17. *The logic* **K4** *does not have a terminating single-conclusion (multi-conclusion) sequent calculus consisting only of single conclusion (multi-conclusion) semi-analytic and context-sharing semi-analytic rules plus some focused axioms.*

Corollary 5.18. *Except the logics* **IPC***,* **LC***,* **KC***,* $\mathbf{Bd_2}$*,* **Sm***,* **GSc** *and* **CPC***, none of the super-intutionistic logics have a terminating single-conclusion sequent calculus consisting only of single conclusion semi-analytic rules and context-sharing semi-analytic rules plus some focused axioms.*

Corollary 5.19. *Except at most six logics, none of the extensions of* **S4** *have a terminating single-conclusion (multi-conclusion) sequent calculus consisting only of single conclusion (multi-conclusion) semi-analytic rules and context-sharing semi-analytic rules plus some focused axioms.*

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